COUNTING COMPONENTS OF PELL-ABEL EQUATIONS WITH GIVEN DEGREE PRIMITIVE SOLUTION.

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The reincarnation of diophantine Pell equation in the realm of polynomials was introduced and investigated by N.H. Abel in 1826 [1]. Since then the equation

$$P^2(x) - D(x)Q^2(x) = 1 \tag{PA}$$

bears the names of both scholars, Pell and Abel. Here $P(x)$ and $Q(x)$ are unknown polynomials of one variable and $D(x) := \prod_{e\in\mathsf{E}}(x-e)$ is a given monic complex polynomial of degree $\deg D = |\mathsf{E}| := 2g+2$ without multiple roots. Generic PA equation admits only the trivial solutions $(P,Q) = (\pm 1, 0)$. For a nontrivial solution to exist certain extra conditions on the coefficient D should be imposed. One form of those conditions has been invented by Abel [1], yet another one will be used below. For a given D, the set of solutions contains a solution with minimal $n := \deg P$, which is called primitive. It generates all the rest solutions P via the composition with the classical Chebyshev polynomials and a change of sign.

The main result of this note consists in finding the number of connected components of complex PA equations with fixed degree $2g+2$ of the polynomial D, which admit a primitive solution P of another fixed degree n.

Theorem 1 *Let $m = min(g, n-g-1)$ and $[\cdot]$ denotes the integer part. PA equation has no primitive solutions of degree $n < g$ or $n > 1$ when $g = 0$. Otherwise, the sought for number of components is equal to $[m/2]+1$ if $n+g$ is odd and $[(m+1)/2]$ if $n+g$ is even.*

We start with the transcendential criterion for the solvability of PA equation [3] in terms of the associated hyperelliptic curve C of genus g: the two point compactification of the affine curve

$$(x, w) \in \mathbb{C}^2 : \qquad w^2 = D(x). \tag{1}$$

Consider a unique differential $d\eta = (x^g + \dots)w^{-1}dx$ on C, with two poles at infinity, residues ± 1 and purely imaginary periods. *PA equation admits a nontrivial solution with deg $P = n$ iff all the periods of $d\eta$ on C lie in the same lattice $2\pi i\mathbb{Z}/n$.* If a PA equation has a nontrivial solution of degree n then the distinguished differential may be represented as $d\eta = n^{-1}d\log(P(x) + wQ(x))$, where from the criterion holds.

A pictorial calculus that allows to effectively control the periods of the distinguished differential in course of the deformation of the curve (1) was designed in [2, 3] for the study of the so called (real) extremal polynomials, where this problem appears too. Quadratic differential $(d\eta)^2$ descends from C to the plane of variable x. Now to each curve (1) we associate a finite planar graph $\Gamma(C)$ built in three steps. STEP 1: Draw all critical vertical trajectories $(d\eta)^2 < 0$ (see [4]) incident to branch points $e \in \mathsf{E}$. STEP 2: Connect all zeros of the differential $d\eta^2$ other than branch points e to the already drawn vertical leaves of the foliation and other such zeros by the horizontal trajectories $d\eta^2 > 0$. Due to the normalization of the distinguished differentials this construction is correct: we obtain finitely many regular analytic arcs. STEP 3: Equip each edge with its length in the metric $|d\eta|$ induced by the distinguished differential.

One of the associated graphs $\Gamma(C)$ that can be obtained in genus 2 is shown on the right of the Fig. 1 up to isotopy of the plane. Here black vertices stand for the branch

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points $e \in \mathsf{E}$ of the curve, white points denote zeros of $(d\eta)^2$, simple lines are horizontal leaves of the foliation and double lines are the vertical ones. For any given genus g there are only finitely many admissible topological types of graphs Γ and they may be listed axiomatically. The graphs $\Gamma(C)$ are completely determined by their properties, so that any topological weighted planar graph meeting five certain requirements stems from a unique (up to inessencial normalization) curve (1) (see [2, 3]). Two of those are: Γ is a tree and the total weight of its vertical edges is π.

A) B)

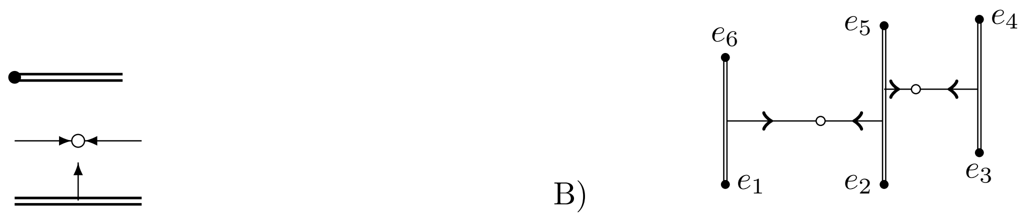

Figure 1: A) Vicinities of nodes for the stable graphs. B) Example of a graph for $g = 2$.

The periods of the distinguished differential of a curve can be reconstructed from a graph: they are integer linear combinations of the weights of the vertical edges of the graph. In particular, one can formulate the local isoperiodic deformations which change the conformal structure of the curve (1) keeping all the periods of $d\eta$ intact. Using isoperiodic graph deformations, any curve C may be brought to another curve with the graph having a standard form. Various standard forms may be used, and we choose stars with vertical edges only with special sequence of weights. The deformation to this form gives us an upper bound for the number of components in the space of PA equation.

To complete the proof of Theorem 1 we need the lower bound for the number of components. We consider graphs Γ embedded in a line. They correspond to multiband real Chebyshev polynomials $P(x)$. Already in this class there are many curves that cannot be isoperiodically transformed one into another. To prove this we use the action of braid group Br_{2g+2} on binary words $(b_1, b_2, \dots) \in (\mathbb{Z}/2\mathbb{Z})^{2g+1}$ defined on the group generators β_s satisfying usual braid relations [5]:

$$\beta_s \cdot (b_1, \dots, b_{s-1}, b_s, b_{s+1}, \dots)^t := (b_1, \dots, b_{s-1} + b_s, b_s, b_{s+1} - b_s, \dots), \qquad s = 1, \dots, 2g+1.$$

This representation is a reduction modulo 2 of a certain specialization of Burau representations of braids [5]. If two curves may be deformed isoperiodically one to the other, the associated periods generate two binary strings in the same orbit of the braid group. It turned out that there are enough orbits of braid action in $(\mathbb{Z}/2\mathbb{Z})^{2g+1}$ for the lower bound on the number of components to be equal to the upper bound.

To conclude we give two applications. First, the moduli space of hyperelliptic curves of genus g with a primitive torsion point of order n has the number of connected components as given in Theorem 1. Second, the moduli space of primitive k-differentials with a unique zero of order $2k$ on curves of genus 2 is connected if $g = 3$ or $g \geq 4$ is even and has two connected components for $k \geq 5$ odd.

We dedicate this paper to J.P.Serre without whom this contribution would never appear.

References

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