DIHEDRAL MONODROMY OF CONE SPHERICAL METRICS

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Abstract. Among metrics of constant positive curvature on a punctured compact Riemann surface with conical singularities at the punctures, dihedral monodromy means that the action of the monodromy group $\mathcal{M} \subset SO(3)$ globally preserves a pair of antipodal points. Using recent results about local invariants of quadratic differentials, we give a complete characterization of the set of conical angles realized by some cone spherical metric with dihedral monodromy.

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1. Introduction

For a compact Riemann surface X of genus g, a finite set of points $A_1, \dots, A_n \in X$ and a distribution of angles $2\pi(a_1, \dots, a_n)$, a natural generalization of uniformization problem is about existence of metric of constant positive curvature in $X \setminus \{A_1, \dots, A_n\}$ that extends to singularities $A_1, \dots, A_n$ as conical points of prescribed angles.

In this problem, Gauss-Bonnet theorem states that the sum of diffuse and singular curvature is a topological invariant. Total diffuse curvature is $2\pi(2g-2+n-\sum a_i)$. Depending on the sign of this quantity, the metric should be hyperbolic, flat or spherical in the complement of the singularities.
If the latter quantity is nonpositive, existence of a hyperbolic or a flat metric has been obtained by several authors, see [12, 13] for recent references. On the opposite, when this quantity is positive, existence of a spherical metric is far from being granted.
Some authors have speculated an equivalence between existence of a metric of constant positive curvature and a notion of stability of bundles (in the spirit of Yau-Tian-Donaldson conjecture for Fano manifolds), see [9].

We could gain a better understanding of cone spherical metrics by the study of metrics with constrained monodromy. In this paper, we focus on a class of metrics introduced in [8]. A cone spherical metric has *dihedral monodromy* if its monodromy group (as a subgroup of SO(3)) globally preserves a pair of antipodal points. This is equivalent to globally preserve the dual great circle. Rotations of the monodromy group are rotations around the axis and

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rotations of angle π around any axis whose antipodal points belong to the preserved great circle (see Section 2.1). A well-studied subclass of metrics with dihedral monodromy is made of metrics with co-axial monodromy, see [1].

In [8], the authors obtain the developing map of a spherical structure with dihedral monodromy by the integration of a quadratic differential. In this way, they prove existence of cone spherical metrics with dihedral monodromy for some distributions of angles. The technical result in their paper (Theorem 1.8) amounts to a characterization of configuration of residues at the poles a quadratic differential may have. They considered only the case of differentials with simple zeroes and double poles.
Their work can be extended and completed. Indeed, in recent papers is given a complete characterization of the configurations of local invariants Abelian and quadratic differentials can realize, see [5, 6]. Using the solution of this problem (Question 1.7 in [8]), we are able to give an explicit characterization of the distributions of angles that can be realized by a cone spherical metric with dihedral monodromy. The key issue, explained in Section 2.6, is the connection between a subclass of spherical metrics (hemispherical surfaces introduced in Definition 2.5) and a subclass of quadratic differentials (totally real Jenkins-Strebel differentials introduced in Definition 2.9).

The main necessary condition for a distribution of angles to be realized by a spherical metric with dihedral monodromy is the *strengthened Gauss-Bonnet inequality* introduced in Proposition 4.2. This inequality is the classical Gauss-Bonnet inequality where only the angles belonging to $\pi\mathbb{Z}$ are considered. We briefly explain why this condition appears. A conical singularity of the spherical metric either corresponds to a double pole of the quadratic differential whose residue is determined by the conical angle or with a singularity of order $k \geq -1$ of the differential. Strengthened Gauss-Bonnet inequality then follows from the classical results about the degree of a divisor associated to a quadratic differential.

There are additional obstructions leading to exceptional distributions of angles that cannot be realized by a spherical metric with dihedral monodromy. They come from general arithmetic obstructions to existence of quadratic differentials with integer residues obtained in [5, 6]. The list of obstructions and the complete characterization of distribution of angles that can be realized are contained in six theorems.
For spherical metrics in genus zero, the classification is given in Theorems 6.3, 6.5 and 6.10 for the strict dihedral case. We remind of [1, Theorem 1] for the co-axial case in Theorem 6.1.
For spherical metrics in higher genus, the classification is given in Theorem 5.2 for the strict dihedral case and Theorem 5.3 for the co-axial case.

The organization of the paper is the following:

- In Section 2, we introduce the co-axial and dihedral monodromy classes. We draw the connection between quadratic differentials and cone spherical metrics with dihedral monodromy.
- In Section 3, we present the results on quadratic and Abelian differentials with prescribed singularities.
- In Section 4, we state the strenghtened Gauss-Bonnet inequality which is the main necessary condition for realization of a distribution of angles by a cone spherical metric with dihedral monodromy.
- In Section 5, we give a characterization of distribution of angles that can be realized in a spherical surface of genus $g \geq 1$ with co-axial or dihedral monodromy (Theorems 5.2 and 5.3), comparing the two classes.

- In Section 6, we state Eremenko's theorem characterizing distribution of angles for cone spherical metrics with co-axial monodromy in genus zero. Comparatively, we state and prove the analogous result for metrics with dihedral monodromy (Theorems 6.3, 6.5 and 6.10).

2. From differentials to spherical metrics with dihedral monodromy

In this section, we recall the basic statements about a special class of spherical surfaces. Then we show how they are related to quadratic differentials and give some background on them.

2.1. **Spherical structures.** A *spherical structure* on a compact surface with finitely many punctures is an atlas of charts with values in the standard sphere $\mathbb{S}^2$ whose transition maps belong to SO(3).

A puncture of a spherical surface is a *conical singularity* of angle θ if it is locally isometric to the singularity of a a hemispherical sector.

Definition 2.1. A *hemispherical sector* of angle $\alpha \in]0;2\pi]$ is a singular surface with boundary, obtained by considering the sector angle α between two meridians in the standard half-sphere and identifying these two sides. It has a conical singularity of angle α and a geodesic boundary of length α.

The definition of hemispherical sectors extends by ramified cover to any angle $\alpha \in \mathbb{R}_+^*$. They provide a geometric model for every conical singularity.

Given a spherical structure on a surface S, there is a representation of the fundamental group of the complement of the punctures in S into the group of linear-fractional transformations. Its image is the *monodromy* of the spherical structure. Note that it is a subgroup of SO(3). A first natural subclass of spherical metrics is given by metrics with *reducible* or *co-axial* monodromy.

Definition 2.2. A cone spherical metric has *co-axial monodromy* if its monodromy group fixes an axis. Equivalently, the monodromy is conjugated to a subgroup of SO(3) conjugated to SO(2).

In this paper, we focus on a class of metrics with slightly more general monodromy, introduced in [8].

Definition 2.3. A cone spherical metric has *dihedral monodromy* if its monodromy group globally preserves a pair of antipodal points. Equivalently, the monodromy is conjugated to a subgroup of SO(3) conjugated to $\mathbb{Z}/2\mathbb{Z} \rtimes \mathrm{SO}(2)$.

Among cone spherical metrics with dihedral monodromy, we distinguish co-axial monodromy (preserving pointwise the two antipodal points of the axis) and strict dihedral monodromy (dihedral but not co-axial).

2.2. **Latitude foliation.** The dihedral monodromy preserves a great circle in the sphere. We will refer to it as the *equator*. We will also refer to the *equatorial net* as the locus in the spherical surface that is mapped to the equator in every chart. Since monodromy acts by isometries, it also preserves the *latitude foliation* that decomposes the sphere into circle of constant latitude. Besides, the *absolute latitude*, which is the absolute value of the angular distance of a point of the sphere to the equator, is also well-defined.

We rephrase the previous paragraph. Given a surface S with a cone spherical metric with dihedral monodromy, the *absolute latitude* is the map $\phi\colon S \to [0, \frac{\pi}{2}]$ which associates to a point the norm of its latitude. The preimage of 0 by ϕ is the *equatorial net* while preimages of $\frac{\pi}{2}$ are the *poles*. As we will see, if the monodromy is in fact co-axial, then the sign of the latitude is globally defined.

It should be noted that the only circle of latitude which is a geodesic is the equator. The others are just loxodromic paths.

Local geometry of conical singularities induces constraints on their position in the latitude foliation. A conical singularity A of angle α should satisfy the following conditions:

- if $\alpha \notin \pi\mathbb{Z}$, then $\phi(A) = \frac{\pi}{2}$;
- if $\alpha \notin 2\pi\mathbb{Z}$, then $\phi(A) \in \{0; \frac{\pi}{2}\}$;

In other words, any singularity can be a pole of the latitude foliation. Otherwise, the angle should be an integer multiple of π corresponding to the number of distinct branches of the foliation that approach the singularity. If this number is odd, then the singularity automatically belongs to the equatorial net. Indeed, the latitude circle the singularity belongs to should be preserved by the nontrivial monodromy of a simple loop around this singularity.

Remark 2.4. For metrics with co-axial monodromy, the *latitude* defined in each chart is preserved by the monodromy and it thus globally defined (not just the absolute latitude). This implies in particular that every singularity whose angle does not belong to $2\pi\mathbb{Z}$ should be a pole of the latitude foliation.

2.3. **Hemispherical surfaces.** They are a special class of spherical surfaces with cones with dihedral monodromy.

Definition 2.5. A *hemispherical surface* is a closed surface with a cone spherical metric obtained by the gluing of finitely many hemispherical sectors along their geodesic boundary. The gluing can identify several boundary points and create conical singularities whose angle is an integer multiple of π.

This class of surfaces is an example of spherical surface with dihedral monodromy.

Proposition 2.6. *A hemispherical surface has dihedral monodromy.*

Proof. For a spherical surface obtained by the gluing of hemispherical sectors, a latitude foliation is defined in each sector and extends globally through the boundary. Monodromy preserves the great circle where the boundary of each sector lies. □

A specific property of hemispherical surfaces is that the latitude of every singularity is either 0 or $\frac{\pi}{2}$.

Hemispherical surfaces are really simple to describe. They are completely characterized by the lengths of the boundary segments and the combinatorics of the gluing pattern. The equatorial net of a hemispherical surface is the union of the boundaries of every cylinder. We show in Subsection 2.4 that any spherical surface with dihedral monodromy can be deformed to a hemispherical surface.

Example 2.7. The most basic hemispherical surface is obtained from a hemispherical sector of angle α. We divide the boundary into two geodesic segments of length $\frac{\alpha}{2}$ and glue them on each other. We obtain a spherical surface with two conical singularities of angle π and one singularity of angle α.

2.4. **Loxodromic projection.** In spherical geometry (and navigation), a *loxodromic path* has constant angle with circles of latitude. In particular, circles of latitude are loxodromic paths.

For any angle θ, we can define the *loxodromic projection flow* L_θ. For any point x of the standard sphere with $\phi(x) \in [0; \frac{\pi}{2}[$, its loxodromic projection $L_\theta(x,t)$ is the intersection of the loxodromic path of constant angle θ passing through x with the circle of latitude $(1-t)\phi(x)$ (with $t \in [0;1]$). In particular, $L_\theta(x,0) = x$ while $L_\theta(x,1)$ belongs to the equator. We define $L_\theta(x,1)$ as the *loxodromic projection* of x to the equator with angle θ.

Proposition 2.8. *For any surface X with a cone spherical metric with dihedral monodromy, there is an isomonodromic deformation $X_{0\leq t\leq 1}$ such that $X = X_0$ and X_1 is a hemispherical surface.*

Proof. The latitude foliation decomposes X into "spherical cylinders" that are continuous families of circles of latitude. For a generic choice of angle θ, the loxodromic projection flow does not make any pair of singularities of theses cylinders collide. In the limit, every "spherical cylinder" which is not bounded by a pole of the latitude foliation degenerates to an interval exchange map. Conical angles have not been modified but the latitude of singularities is then either 0 or $\frac{\pi}{2}$. □

2.5. **Quadratic differentials and half-translation surfaces.** On a Riemann surface X, a quadratic differential q is a meromorphic section of $K_X^{\otimes 2}$. Outside singularities of odd order, it writes locally as the square of a meromorphic 1-form $\pm\sqrt{q}$.
Antiderivatives of $\pm\sqrt{q}$ are the developing maps of a *half-translation structure*. This structure is formed by an atlas in the complement of the singularities of q. This atlas is made of charts to $\mathbb{C}$ whose transition maps are of the form $z \mapsto \pm z + c$.

A singularity of order $a \geq -1$ corresponds to a conical singularity of angle $(2+a)\pi$ in the associated flat metric. A double pole is a point at infinity in an infinite cylinder. Besides, at a double pole, if $q = (\frac{r_{-2}}{z^2} + \frac{r_{-1}}{z} + r_0 + r_1 z + \dots)dz^2$, then the quadratic residue at the double pole is r_{-2}. It should always be nonzero. In the cylinder neighboring the double pole of quadratic residue r, the flat cylinder is obtained by identifying the two sides of an half-infinite band by translation of $\pm\sqrt{r}$, see [10] for details.

We define a special class of quadratic differentials.

Definition 2.9. A quadratic differential q on a Riemann surface X is a *totally real Jenkins-Strebel differential* if its associated half-translation surface is formed by the gluing of finitely many semi-infinite cylinders along segments of their boundary.

Recall that a *period* of q is the integral of $\pm\sqrt{q}$ along a path between two singularities of order ≥ -1. A period is *absolute* if both starting and ending points coincide. The set of absolute periods forms a subgroup of $\mathbb{C}$. Note that the periods of a totally real Jenkins-Strebel differential are real. It has exactly one double pole for each cylinder and the other singularities are conical singularities of angle in $\pi\mathbb{Z}$. They are either simple poles or zeroes of the differential.

2.6. **Relation between hemispherical surfaces and totally real Jenkins-Strebel differentials.** There is a construction which associates to any hemispherical surface S a flat surface S_{flat}. It replaces every hemispherical sector of angle $2a\pi$ by a semi-infinite cylinder of circumference a. The lengths of the segments in the boundary are preserved (up to a factor 2π).

Conversely, given a half-translation surface defined by a totally real Jenkins-Strebel differential, we replace each semi-infinite cylinder by q hemispherical sector. This operation is the inverse of the previous one.

The group of absolute periods of the quadratic differential is the image of the monodromy group of the hemispherical surface into the rotation group along the preserved axis (up to a factor 2π).

A totally real Jenkins-Strebel differential q is the global square of 1-form if and only if every cylinder of the flat surface can be given a sign such that every boundary segment bounds a positive and a negative cylinder.
The latter condition implies in particular that singularities should be of even order (with an angle in $2\pi\mathbb{Z}$).
Translated in the language of spherical geometry, a hemispherical surface has globally defined latitude if and only if its totally real Jenkins-Strebel differential is the global square of a 1-form. This property is equivalent to co-axial monodromy.

2.7. **Some background on complex projective structures and quadratic differentials.** In this section, we recall some facts about complex projective structures and their relation with quadratic differentials. This gives interesting background on this work but will not be used in the remaining sections.

A *complex projective structure* is defined by charts with values in $\mathbb{CP}^1$ and transition maps in $\mathrm{PSL}(2,\mathbb{C})$. In particular, a spherical structure is a complex projective structure whose monodromy is conjugated to a subgroup of $\mathrm{SO}(3)$.

A complex projective structure has *dihedral monodromy* if it globally preserves a pair of points of $\mathbb{CP}^1$, see Section 6 in [4]. Up to conjugation, we can assume these preserved points are $\{0,\infty\}$. Then the monodromy will act by functions of the form $z \mapsto az^{\pm 1}$ with $a \in \mathbb{C}^*$.
This directly implies existence of a quadratic differential q such that any developing map of the structure is of the form $e^{\int^z \sqrt{-q}}$. Besides, in order to avoid irregular singularities of the developing map, it is reasonable to restrict to quadratic differentials with at worse double poles (logarithmic singularities).

In this class of projective structures, the monodromy preserves a metric if and only if $a \in \mathrm{U}(1)$ for the transition maps. In other words, periods of the quadratic differential should be real (see Section 1.1 of [8] for details). In other words, the quadratic differential should be Jenkins-Strebel.

Then, totally real Jenkins-Strebel quadratic differential introduced in definition 2.9 are those for which singularities either belong to the equator or the poles of the latitude foliation. The whole correspondence is summarised in Table 1.

Complex-analytic object	Geometric structure
Quadratic differential	Complex projective structure with dihedral monodromy
Jenkins-Strebel quadratic differential	Cone spherical metric with dihedral monodromy
Totally real Jenkins-Strebel differential	Hemispherical surface

Table 1. Correspondence between analytic and geometric objects.

3. Differentials with prescribed singularities

In this section, we recall some results of [5, 6] about existence of differentials with prescribed local behaviour. Moreover, we recall an important operation on differentials called the *contraction flow*.

3.1. **Quadratic differentials.** By abuse of notation, we refer to a pair (X, q) where X is a Riemann surface of genus g and q a quadratic differential on X as a quadratic differential. The moduli spaces of these objects are stratified according to the orders of the singularities of q. In this paper, quadratic differentials have at worse double poles. For $a_1, \dots, a_n \in 2\mathbb{N}^*$ and $b_1, \dots, b_m \in 2\mathbb{N} - 1$, the *stratum of quadratic differentials* with zeroes and simple poles of multiplicities $a_1, \dots, a_n, b_1, \dots, b_m$ and p double poles is denoted $\mathcal{Q}(a_1, \dots, a_n, b_1, \dots, b_m, -2^p)$. See [7] for details.

We also require that quadratic differentials are not global squares of 1-forms. Such differentials are called *primitive*. Riemann-Roch theorem (or Gauss-Bonnet theorem) implies that $\sum a_i + \sum b_j - 2p = 4g - 4$.

For such quadratic differentials, classical complex analysis shows that the only local invariants up to biholomorphism are the order of the singularity and the quadratic residue in the case of a double pole, see [10] for details. At a double pole, every quadratic differential is (up to a biholomorphic change of variables) $\left(\frac{rdz}{z}\right)^2$, where r^2 is the quadratic residue.

In Theorems 1.1, 1.2, 1.3 and 1.9 of [6] has been proved a complete characterization of configurations of quadratic residues that can be realized in any given stratum of quadratic differentials. We extract from these results the following two theorems.

Theorem 3.1. *Let $\mathcal{Q}$ be a stratum $\mathcal{Q}(a_1, \dots, a_n, b_1, \dots, b_m, -2^p)$ of primitive quadratic differentials on a surface of genus $g \geq 1$ such that $p \geq 1$.*
Every configuration of quadratic residues $(r_1, \dots, r_p) \in (\mathbb{R}_+^*)^p$ is realized by a quadratic differential of $\mathcal{Q}$ with the exception of configurations $(r, \dots, r)$ for strata $\mathcal{Q}(4s, -2^{2s})$ and $\mathcal{Q}(2s+1, 2s-1, -2^{2s})$ in genus one.

In genus zero, it should be noted that a quadratic differential is primitive if and only if at least one of its singularities is of odd order.

Theorem 3.2. *Let $\mathcal{Q}$ be a stratum $\mathcal{Q}(a_1, \dots, a_n, b_1, \dots, b_m, -2^p)$ of primitive quadratic differentials on the Riemann sphere with $m, p \geq 1$.*
Every configuration of quadratic residues $(b_1, \dots, b_p) \in (\mathbb{R}_+^*)^p$ is realized by a quadratic differential of $\mathcal{Q}$ with the following exceptions:

- *$\mathcal{Q}(p-2, p-2, -2^p)$ with p odd and configurations of the form $(A^2, B^2, C^2, \dots, C^2)$ with $C = A + B$ or $B = A + C$ and $A, B, C > 0$;*
- *$\mathcal{Q}(p-1, p-3, -2^p)$ with p even and configurations of the form $(A^2, A^2, B^2, \dots, B^2)$ with $A, B > 0$;*
- *$\mathcal{Q}(a_1, \dots, a_n, b_1, b_2, -2^p)$ and configurations of the form $(L \cdot f_1^2, \dots, L \cdot f_p^2)$ with $L > 0$, $f_1, \dots, f_p \in \mathbb{N}^*$, $\sum f_j$ is even and $\sum f_j < 2p$;*
- *$\mathcal{Q}(a_1, \dots, a_n, b_1, b_2, -2^p)$ and configurations of the form $(L \cdot f_1^2, \dots, L \cdot f_p^2)$ with $L > 0$, $f_1, \dots, f_p \in \mathbb{N}^*$, $\sum f_j$ is odd and $\sum f_j \leq \max(b_1, b_2)$.*

3.2. **Abelian differentials.** Some quadratic differentials are global squares of Abelian differentials (1-forms). Therefore, square roots of quadratic residues are globally defined (up to a common factor). Obstructions to realization of configurations of residues in any stratum states differently.

We consider the *stratum* $\mathcal{H}(a_1,\dots,a_n,-1^p)$ of pairs (X,ω) where X is a compact Riemann of genus g and ω is a meromorphic 1-form with zeroes of orders $a_1,\dots,a_n$ and p simple poles. Gauss-Bonnet theorem implies that $\sum a_i - p = 2g-2$.
We consider real configurations of reals $(\lambda_1,\dots,\lambda_x,-\mu_1,\dots,-\mu_y)$ with $x+y=p$, whose sum is zero (to satisfy Residue theorem) and such that $\lambda_1,\dots,\lambda_x,\mu_1,\dots,\mu_y>0$.

In [5], we show in Theorem 1.1 that in genus at least one, the only obstruction is the Residue theorem.

Theorem 3.3. *In any stratum $\mathcal{H}(a_1,\dots,a_n,-1^p)$ of meromorphic 1-forms on a surface of genus $g\geq 1$ with $p\geq 1$, every configuration of residues $(\lambda_1,\dots,\lambda_x,-\mu_1,\dots,-\mu_y)$ with $\sum\lambda_i=\sum\mu_j$ is realized by a differential in the stratum.*

In genus zero, there is an additional arithmetic obstruction, described in Theorem 1.2 of [5] and Theorem 2 of [1].

Theorem 3.4. *In any stratum $\mathcal{H}(a_1,\dots,a_n,-1^p)$ of meromorphic 1-forms on a surface of genus zero with $p\geq 1$, every configuration of residues $(\lambda_1,\dots,\lambda_x,-\mu_1,\dots,-\mu_y)$ with $\sum\lambda_i=\sum\mu_j$ is realized by a differential in the stratum with the following exception.*
If the configuration of residues is of the form $(L\cdot f_1,\dots,L\cdot f_x,-L.g_1,\dots,-L.g_y)$ with $L>0$ and $f_1,\dots,f_x,g_1,\dots,g_y$ are integers without nontrivial common factor, then it can be realized in the stratum if and only if $\sum f_i=\sum g_j>\max(a_1,\dots,a_n)$.

3.3. **Contraction flow.** In Sections 3.1 and 3.2, we considered quadratic differentials defining flat surfaces of infinite area (because cylinders around double poles are semi-infinite). There is an action of $\mathrm{GL}^+(2,\mathbb{R})$ on strata of quadratic differentials. This group acts by postcomposition in the charts and acts naturally on the periods, see [14] for details.

The *contraction flow* is a one-parameter flow in $\mathrm{GL}^+(2,\mathbb{R})$ that preserves a direction and contracts exponentially another. For a quadratic differential defining a flat surface of infinite area, if the contraction flow contracting a generic direction (a direction where there is no saddle connection), then it converges to a surface in the stratum where every (absolute and relative) period belongs to the preserved direction, see Section 5.4 of [11].
Infinite area hypothesis is necessary because otherwise the area of the flat surface would shrink along the flow.

If we apply the contraction flow to any differential that realizes a real configurations of residues (like in Sections 3.1 and 3.2), a generic contraction flow preserving the horizontal direction will converge to a *totally real Jenkins-Strebel differential*.

Corollary 3.5. *If a distribution of angles is realized by a cone spherical metric with dihedral monodromy, it is realized by a hemispherical surface with the same monodromy.*

Remark 3.6. The loxodromic projection defined in Section 2.3 in the spherical counterpart of the contraction flow.

4. Strengthened Gauss-Bonnet inequality

For a distribution of angles and a given genus, we would like to know if the distribution of angles is realized by a cone spherical metric with dihedral monodromy on a surface of genus g. Conical singularities with integer or half-integer angles play a special role.

We distinguish between:

- *even* conical singularities for which the angle is in $2\pi\mathbb{Z}$;
- *odd* conical singularities for which the angle is in $\pi(2\mathbb{Z}+1)$;
- *non-integer* conical singularities for which the angle is not in $\pi\mathbb{Z}$.

For a cone spherical metric with n conical singularities, we have $n = n_E + n_O + n_N$. These three terms are respectively the numbers of even, odd and non-integer conical singularities.

Definition 4.1. We consider distributions of angles $2\pi(a_1,\dots,a_{n_E},b_1,\dots,b_{n_O},c_1,\dots,c_{n_N})$ where the three subfamilies are respectively even ($a_i \in \mathbb{Z}$), odd ($b_i \in \mathbb{Z}+\frac{1}{2}$) and non-integer angles ($c_i \notin \frac{1}{2}\mathbb{Z}$). We define:

- the *total sum* $\sigma = \sum a_i + \sum b_j + \sum c_k$;
- the *maximal integral sum* T as the sum of $\sum a_i$ and the $2\lfloor\frac{n_O}{2}\rfloor$ biggest numbers among the b_j (in particular, $T \in \mathbb{Z}$).

Obviously, we have $\sigma \geq T$. If there is an even number of odd singularities, then $T = \sum a_i + \sum b_j$.

Just like a Gauss-Bonnet inequality is necessary for realization of a distribution of angles by a cone spherical metric, a necessary condition for realization by a spherical metric with a dihedral monodromy is that a Gauss-Bonnet inequality should be fulfilled by the integer singularities alone.

Proposition 4.2. *Let $2\pi(a_1,\dots,a_{n_E},b_1,\dots,b_{n_O},c_1,\dots,c_{n_N})$ be a distribution of angles. If this distribution of angles is realized by a cone spherical metric with dihedral monodromy on a surface of genus g, then it should satisfy strengthened Gauss-Bonnet inequalities:*

- $T \geq 2g+n-1$ *if n_O is even and $n_N = 0$;*
- $T \geq 2g+n-2$ *otherwise.*

Proof. Following Section 2.3, if such a distribution of angles is realized by a cone spherical metric with dihedral monodromy, then it can be realized by a hemispherical surface S. In this hemispherical surface, singularities with non-integer angles are automatically at the poles of the latitude foliation.
Among even and odd singularities, some belong to the equatorial net while some other are at the poles. If the monodromy is dihedral, then every odd singularity is also at the pole. We consider the totally real Jenkins-Strebel differential q corresponding to hemispherical surface S.
Quadratic differential q belongs to a stratum $\mathcal{Q}(d_1,\dots,d_s,-2^t)$ with conical singularities of orders $d_1,\dots,d_s$ and t double poles. We automatically have $d_1+\dots+d_s = 4g-4+2t$.
Since an even or an odd singularity is either a conical singularity of q or a double pole, we have $t \geq n_N + (n_E+n_O-s)$. Consequently, $s+t \geq n$ and $d_1+\dots+d_s+2s = 4g-4+2t+2s \geq 4g+2n-4$.
Since every other singularity is a double pole, $d_1+\dots+d_s$ is even and $\sum(d_i-2) \leq T$. This implies $T \geq 2g+n-2$.
If n_O is even, $n_N = 0$ and $T = 2g+n-2$, then $\sigma = T$ and the classical Gauss-Bonnet inequality requires $\sigma > 2g+n-2$. □

5. Spherical structures in higher genus

We apply the results about quadratic and Abelian differentials with prescribed residues in order to construct hemispherical surfaces realizing the adequate distribution of angles.

Using equivalence stating in Section 2.4, we construct hemispherical surfaces corresponding to totally real Jenkins-Strebel differentials. It should be noted that in genus at least one, in strata of quadratic and Abelian differentials, there are several degrees of freedom in addition to the residues. In other words, if there is a differential that realize some configuration of real residues, then there is another differential that realizes it and such that the group of periods of the differential is dense in $\mathbb{R}$. For such a totally real Jenkins-Strebel differential, the monodromy of the corresponding hemispherical surface is dense in the subgroup of SO(3) preserving a pair of antipodal points (pointwise in the co-axial case and globally in the dihedral case).

5.1. **Strict dihedral case.** Specific obstruction in genus one for quadratic differentials with prescribed residues (Theorem 3.1) lead to four exceptional families of distributions of angles that cannot appear in spherical surfaces with strict dihedral monodromy.

Proposition 5.1. *Four exceptional families of distributions of angles cannot be realized by a cone spherical metric with strict dihedral monodromy on a torus:*

- $(4k+2)\pi, c, \dots, c$ *with* $2k$ *non-integer equal angles* $c \notin \pi\mathbb{Z}$*;*
- $(2k+3)\pi, (2k+1)\pi, c, \dots, c$ *with* $2k$ *non-integer equal angles* $c \notin \pi\mathbb{Z}$*;*
- $(4k+2)\pi$ *for any integer* k*;*
- $(2k+3)\pi, (2k+1)\pi$ *for any integer* k*.*

Proof. According to Proposition 2.8, if such a distribution of angles is realized by a cone spherical metric with dihedral monodromy, then it can be done by a hemispherical surface. In this hemispherical surface, singularities with non-integer equal angles are automatically at the poles of the latitude foliation.
Following Section 2.6, a hemispherical surface corresponds to a totally real Jenkins-Strebel differential. Zeroes and simple poles of the quadratic differential can only be the conical singularities with integer angle. Besides, since the other singularities are double poles, the sum of their orders should be even. Consequently, these (primitive) quadratic differentials are in $\mathcal{Q}(4k, -2^{2k})$ or $\mathcal{Q}(2k+1, 2k-1, -2^{2k})$. In both cases, Theorem 3.1 implies that in these strata, a configuration of uniformly equal residues cannot be realized. If the distribution of angles contains $2k$ equal non-integer angles c, the configuration $c, \dots, c$ cannot be realized. If there is no non-integer angle, then, the $2k$ double poles of the quadratic differentials correspond to regular points of the spherical metric and their quadratic residue is 1. In this case, it cannot be realized either. □

Outside these four exceptional families, the only condition that should be satisfied is strengthened Gauss-Bonnet inequality stated in Proposition 4.2.

Theorem 5.2. *Let* $2\pi(a_1, \dots, a_{n_E}, b_1, \dots, b_{n_O}, c_1, \dots, c_{n_N})$ *be a distribution of angles as in Section 4. Outside obstructions in genus 1 described in Proposition 5.1, there exists a cone spherical metric with strict dihedral monodromy on a surface of genus* $g \geq 1$ *with* n *conical singularities of prescribed angles if and only if strengthened Gauss-Bonnet inequality is satisfied.*

Proof. We assume the distribution of angles satisfies strengthened Gauss-Bonnet inequality (Proposition 4.2) and is not concerned by obstructions of Proposition 5.1. We are going to prove existence of a cone spherical metric with strict dihedral monodromy in the category of hemispherical surfaces.
Since strengthened Gauss-Bonnet inequality is satisfied, we can find in the distribution of angles a subset of s even or odd singularities of angles $\pi(2+d_i)$ such that $d_1 + \dots + d_s$ is even number whose value is at least $4g - 4 + 2(n-s)$ (or strictly bigger if n_O is even and $n_N = 0$). Then, we work with a nonempty stratum $\mathcal{Q}(d_1, \dots, d_s, -2^t)$ of primitive quadratic differentials on Riemann surfaces of genus g.
If $g \geq 2$, any configuration of real positive quadratic residues are realized in the stratum

(Theorem 3.1) and they are also realized by totally real Jenkins-Strebel differential (see Section 3.3). Therefore, we can construct the analogous hemispherical surface (see Section 2.4). If $g = 1$, unless the maximal integral sum (see Definition 4.1) is realized by angles $(4k+2)\pi$ or $(2k+3)\pi, (2k+1)\pi$, the same construction can be carried out (indeed, in Theorem 3.1, the only strata with obstructions are $\mathcal{Q}(4k, -2^{2k})$ and $\mathcal{Q}(2k+1, 2k-1, -2^{2k})$). In the remaining cases, the only configurations of residues that cannot be realized are formed by an even number of identical residues. Consequently, the remaining conical singularities of the spherical structure that correspond to double poles of the quadratic differentials should have equal angles. This rules out the possibility of odd singularities among them. There are thus two possibilities. Either every double pole has a quadratic residue equal to 1 (because it corresponds to a regular point of the spherical metric) or they all correspond to non-integer singularities. These cases are covered by Proposition 5.1. □

5.2. **Co-axial case**. In genus at least one, there is no obstruction to existence of Abelian differentials with prescribed singularities (outside residue theorem). We consider distributions of angles $\pi(a_1, \dots, a_n, c_1, \dots, c_p)$ with $a_1, \dots, a_n \in 2\mathbb{N}+2$ and $c_1, \dots, c_p \notin 2\mathbb{N}+2$.

The following result is analogous to Theorem 1 of [1] concerning spherical metrics with co-axial monodromy on punctured spheres.

Theorem 5.3. *Let $2\pi(a_1, \dots, a_n, c_1, \dots, c_p)$ be a distribution of angles. There exists a spherical metric with co-axial monodromy on a surface of genus $g \geq 1$ with $n+p$ conical singularities of prescribed angles if and only if:*

- *there is a sequence of signs $\epsilon_1, \dots, \epsilon_p$ such that $K = \sum_{j=1}^{p} \epsilon_j c_j \in \mathbb{N}$;*
- *$\sum a_i - 2g + 2 - n - p - K$ is nonnegative and even if $p \geq 1$;*
- *$\sum a_i - 2g + 2 - n$ is positive and even if $p = 0$.*

Proof. A distribution of angles is realized by a metric with co-axial monodromy if and only if it is realized by some hemispherical surface (see Proposition 2.8). The totally real Jenkins-Strebel differential corresponding to such a hemispherical surface is the square of an Abelian differential (see Sections 2.5 and 2.6).
Therefore, there is a subset of $(a_1, \dots, a_n)$ corresponding to singularities that will belong to the equatorial net (and will be zeroes of the differential). Without loss of generality, we assume they are the s first singularities of the list. Then, the simple poles of the differential will be the $n-s$ other integer singularities, the p non-integer singularities and some additional regular points (simple poles with residues ± 1). Each of them should be given a sign in such a way they sum to zero (and satisfy residue theorem). Since there is no obstruction in genus at least one (Theorem 3.3), we only have to check stratum is nonempty.
If a distribution of angles is realized in such a way an integer singularity of angle $2a\pi$ counts as a simple pole of the Abelian differential (of residue $\pm a$), then it can also be realized in such a way this singularity counts as a zero of the form. Indeed, starting from a configuration of residues summing to zero realized in stratum $\mathcal{H}$, we consider stratum $\mathcal{H}'$ with $a-1$ additional simple poles and an additional zero of order $a-1$. We replace the simple pole of residue a by a simple poles of residue 1 (up to a change of sign). Consequently, we just have to consider hemispherical surfaces where every integer singularity counts as a zero of the Abelian differential.
We need to find an Abelian differential in stratum $\mathcal{H}(a_1 - 1, \dots, a_n - 1, -1^t)$ where $t = \sum a_i - n - 2 - 2g$. Besides, simple poles are of residues ± 1 or $\pm c_i$. Since the total residue is zero, there is a signed sum of $(c_1, \dots, c_p)$ which is an integer K. If $p = 0$, we just need to have an even number of simple poles (one half having residue 1 while the other half has residue -1). The condition amounts to $t = \sum a_i - n - 2 - 2g$ being even and positive.
If $p \geq 1$, then $t - p$ should be positive and its value should be at least K with the same parity (if it is bigger, it should be bigger by an even number in order to keep a total residue

equal to zero with residues compensating each other). Following Theorem 3.3, there is no obstructions to realize differentials with prescribed residues in genus at least one so the condition described above are necessary and sufficient. □

5.3. **Comparison.** In genus at least one, almost every distribution of angles that are realized with co-axial monodromy can also be realized with strict dihedral monodromy. The exceptions are two of the four families of Proposition 5.1.

Proposition 5.4. *Let $2\pi(a_1, \dots, a_n, c_1, \dots, c_p)$ be a distribution of angles. If it is realized by a spherical metric with co-axial monodromy on a surface of genus g, then it can also be realized by a metric with strict dihedral monodromy on a surface of genus g with the following the following two families of exceptions in genus one (parametrized by $k \in \mathbb{N}^*$):*

- *$(4k+2)\pi, c, \dots, c$ with $2k$ equal angles $c \notin \pi\mathbb{Z}$;*
- *$(4k+2)\pi$.*

The latter present obstruction for dihedral monodromy (Proposition 5.1) but clearly satisfy hypothesis of Theorem 5.3.

Proof. Since the distribution is realized by a metric with co-axial monodromy, we have $\sum a_i \geq 2g - 2 + n + p$. If $g \geq 2$ and $p \geq 1$, this implies existence of a metric with strict dihedral monodromy (Theorem 5.2).
If $g \geq 2$ but $p = 0$, classical Gauss-Bonnet implies $\sum a_i > 2g-2+n$ and thus $\sum a_i \geq 2g+n-1$. This also implies existence of the adequate metric.
Then we consider the genus one case. If $p = 0$, then we have $\sum a_i > n$. Theorem 5.2 then implies existence of metric with strict dihedral monodromy except in the four exceptional cases of Proposition 5.1. Among them, the only family for which there is obstruction is where there is only one conical singularity the angle of which is of the form $(4k+2)\pi$.
Then, we assume $g = 1$ and $p \geq 1$. We have $\sum a_i \geq 2g - 2 + n + p$. Therefore, strengthened Gauss-Bonnet inequality is satisfied and Theorem 5.2 implies the existence of a metric with strict dihedral monodromy except in the cases of Proposition 5.1. Since there is at least one even singularity and $p \geq 1$, we just have to avoid the case where the distribution of angles is $(4k+2)\pi, c, \dots, c$ with $2k$ non-integer equal angles $c \notin \pi\mathbb{Z}$. □

6. Spherical structures in genus zero

In punctured spheres, monodromy of cone spherical metrics is generated by rotations around singularities. In particular, if every singularity is even (its angle belongs to $2\pi\mathbb{Z}$), then the monodromy of the metric is trivial and the geometric structure is just a cover of the sphere ramified at these singularities. This case has been already classified, see Section 7 in [2].

In the following, we assume at least one conical singularity has a non-integer angle. If there is only one such singularity, then monodromy is cyclic and thus automatically co-axial. In the next subsection, we will see that in fact, this implies existence of a second conical singularity with a non-integer angle.

6.1. **Co-axial case.** In this section, we consider geometric structures whose monodromy group is contained in the group of rotations around an axis. In particular, it could be finite rotation group (if angles of non-integer singularities belong to $\pi\mathbb{Q}$ for example).

Theorem 1 in [1] gave a complete classification of distribution of angles that are realized with a co-axial monodromy.

Theorem 6.1. *Let $2\pi(a_1, \dots, a_n, c_1, \dots, c_p)$ be a distribution of angles with $p \geq 1$. There exists a spherical metric with co-axial monodromy on a punctured sphere with $n + p$ conical singularities of prescribed angles if and only if:*

- *there is a sequence of signs $\epsilon_1, \ldots, \epsilon_p$ such that $K = \sum_{j=1}^{p} \epsilon_j c_j \in \mathbb{N}$;*
- *$M = \sum a_i + 2 - n - p - K$ is nonnegative and even;*
- *if $c_1, \ldots, c_p$ are commensurable, an additional arithmetic condition should hold.*

Let v be the vector $(c_1, \ldots, c_p, 1, \ldots, 1)$ with $M + K$ elements equal to 1. If v is of the form $L(b_1, \ldots, b_{p+M+K})$ with $L > 0$ and $b_1, \ldots, b_{p+M+K}$ are integers, then the additional condition is:

$$2 \max(a_1, \ldots, a_n) \leq \sum b_i.$$

The condition about signed sums of non-integer angles means in particular that there should be at least two non-integer singularities.

A proof analogous to the one of Theorem 5.3 could be given to Theorem 6.1 by using Theorem 3.4.

6.2. **Strict dihedral case.** In the problem of realization of distribution of angles by a metric with strict dihedral monodromy, the nature of obstructions depends crucially on the number of odd singularities.

We first give a restriction on the number of odd singularities.

Lemma 6.2. *Let $2\pi(a_1, \ldots, a_{n_E}, b_1, \ldots, b_{n_O}, c_1, \ldots, c_{n_N})$ be a distribution of angles as in Section 4. If it is realized by a cone spherical metric with strict dihedral monodromy on the punctured sphere, then $n_O \geq 2$.*

Proof. If the distribution is realized, then it can be realized by a hemispherical surface (Proposition 2.8). The latter corresponds to a primitive quadratic differential (Section 2.5). In genus zero, primitive quadratic differentials have singularities of odd order (otherwise they are global squares of 1-forms). Besides, they have an even number of singularities of odd order (since the total order is -4). Consequently, there are at least two odd singularities belonging to the equatorial net of the hemispherical surface. □

We now treat the cases according to the number of odd singularities. We begin with the case with at least 4 odd singularities, then we treat the case with 3 odd singularities and finally with 2.

6.2.1. *At least four odd singularities.* In the case $n_O \geq 4$, there is no additional obstruction to strengthened Gauss-Bonnet inequality.

Theorem 6.3. *Let $2\pi(a_1, \ldots, a_{n_E}, b_1, \ldots, b_{n_O}, c_1, \ldots, c_{n_N})$ be a distribution of angles as in Section 4. If $n_O \geq 4$, there exists a cone spherical metric with strict dihedral monodromy on a punctured sphere with n conical singularities of prescribed angles if and only if strengthened Gauss-Bonnet inequality is satisfied. Besides, the metric can be chosen in such a way its monodromy group is infinite.*

Proof. We assume the distribution of angles satisfies strengthened Gauss-Bonnet inequality (Proposition 4.2). We prove existence of a cone spherical metric with strict dihedral monodromy in the category of hemispherical surfaces.

Since strengthened Gauss-Bonnet inequality is satisfied, we can find in the distribution of angles a subset of s even or odd singularities of angles $\pi(2 + d_i)$ such that $d_1 + \cdots + d_s$ is even number whose value is at least $4g - 4 + 2(n - s)$ (or strictly bigger if n_O is even and $n_N = 0$). Besides, we assume there are at least four odd singularities among the s chosen singularities. Then, we consider a nonempty stratum $\mathcal{Q}(d_1, \ldots, d_s, -2^t)$ of primitive quadratic differentials on the Riemann sphere. There are at least four singularities of odd order in quadratic differentials of these strata. Therefore, any configuration of real positive quadratic residues are realized in the stratum (Theorem 3.2) and they are also realized by totally real Jenkins-Strebel differential (see Section 3.3). Therefore, we can construct the

analogous hemispherical surface (see Section 2.4).
Finally, we have to prove that we can realize the hemispherical surface in such a way the projection of the monodromy group on the rotation group around the preserved axis has dense image. For a quadratic differential on the Riemann sphere with at least four odd singularities, the canonical double cover ramified at the odd singularities is of genus at least one (Riemann-Hurwitz formula). Therefore, there are other degrees of freedom than quadratic residues. Up to a small perturbation, we can assume the group of absolute periods of the Abelian differential in the cover is dense in $\mathbb{R}$. □

6.2.2. *Three odd singularities.* If $n_O = 3$, there are specific obstructions that require to be handled separately.

Proposition 6.4. *Let $k \in \mathbb{N}$, $l \geq k$ and $\alpha, \beta \notin \pi\mathbb{Z}$. If*

$$\alpha + \beta = (2l+1)\pi \text{ or } \alpha + (2l+1)\pi = \beta \text{ or } \beta + (2l+1)\pi = \alpha\,,$$

then distribution of angles $((2k+1)\pi, (2k+1)\pi, (2l+1)\pi, \alpha, \dots, \alpha, \beta)$ with $2k-1$ angles equal to α cannot be realized by a cone spherical metric with strict dihedral monodromy on a punctured sphere.

Proof. If such a distribution is realized by a cone spherical metric, then it is also realized by a hemispherical surface. The latter should have at least two half-integer singularities on its equatorial net. This implies existence of a quadratic differential in $\mathcal{Q}(2k-1, 2k-1, -2^{2k+1})$ whose quadratic residues are $\left(l+\frac{1}{2}\right)^2, \left(\frac{\alpha}{2\pi}\right)^2$ and $2k-1$ quadratic residues equal to $\left(\frac{\beta}{2\pi}\right)^2$. This configuration is forbidden in Theorem 3.2. □

In addition to strengthened Gauss-Bonnet inequality stated in Proposition 4.2 and obstruction of Proposition 6.4, an arithmetic condition should be satisfied if every conical singularity has a rational angle.

Theorem 6.5. *Let $2\pi(a_1, \dots, a_{n_E}, b_1, b_2, b_3, c_1, \dots, c_{n_N})$ be a distribution of angles. We assume $b_1 \geq b_2 \geq b_3$. Outside obstructions described in Proposition 6.4, there exists a cone spherical metric with strict dihedral monodromy on a punctured sphere with n conical singularities of prescribed angles if and only if:*

- *strengthened Gauss-Bonnet inequality $T = \sum a_i + b_1 + b_2 \geq n-2$ holds;*
- *an additional arithmetic condition described below is satisfied when $c_1, \dots, c_{n_N} \in \pi\mathbb{Q}$.*

If vector $(c_1, \dots, c_{n_N}, b_3, 1, \dots, 1)$ with $T+2-n$ elements equal to 1 is of the form $L(r_1, \dots, r_{T-n_E})$ with $L > 0$ and $r_1, \dots, r_{T-n_E}$ integers, then:

- *$\sum r_i \geq b_1 + b_2$, if $\sum r_i$ is even;*
- *$\sum r_i \geq b_1$, if $\sum r_i$ is odd.*

Proof. We assume the distribution of angles satisfies strengthened Gauss-Bonnet inequality (Proposition 4.2) and is not concerned by obstructions of Proposition 6.4. We are going to prove existence of a cone spherical metric with strict dihedral monodromy in the category of hemispherical surfaces.
Since strengthened Gauss-Bonnet inequality is satisfied, we can find in the distribution of angles a subset of s even or odd singularities of angles $\pi(2+d_i)$ such that $d_1 + \dots + d_s$ is an even number whose value is at least $4g-4+2(n-s)$. Among them there should be two of the three odd singularities. Then, we consider a nonempty stratum $\mathcal{Q}(d_1, \dots, d_s, -2^t)$ of primitive quadratic differentials on the Riemann sphere.
Then, we have to check if the configuration of residues determined by the remaining singularities and the additional double poles is realized in the stratum. If one the four obstructions of Theorem 3.2 holds and one even singularity of angle $2a_i\pi$ is not among the s chosen singularities, then if we can just take stratum $\mathcal{Q}(2a_i-2, d_1, \dots, d_n, -2^{t+a_i-1})$. The first two obstructions do not hold when there is zero of even order in differentials of the stratum.

Concerning the two other obstructions, we add an integer residue to a configuration in which there is a residue which is the square of an element of $\mathbb{N} + \frac{1}{2}$. Therefore, the sum of integer weights increases by at least two. If the initial configuration is not realized in its stratum, it cannot be worse with the new one. Consequently, we assume every even singularity is among the s chosen singularities.
Similarly, we prove that we could have chosen b_1 and b_2 for the two odd singularities that will correspond to conical singularities of the flat metric induced by the quadratic differential. For the two last obstructions, replacing an odd singularity by one with a bigger order, we add at least one double pole with quadratic residue equal to 1, increasing the sum of integer weights by at least two. The two first obstructions appear only if $n_E = 0$. The second one involves a configuration of quadratic residues of the form $(A^2, A^2, B^2, \dots, B^2)$ with $A, B > 0$ and an even number of B^2. Here, the third odd singularity corresponds to a double pole with a quadratic residue equal to b_j^2. There is no double pole with the same quadratic residue otherwise there would be a fourth odd singularity in the angle distribution. Therefore, the second obstruction is not relevant when $n_O = 3$. The first obstruction of Theorem 3.2 is about configurations of quadratic residues of the form $(A^2, B^2, C^2, \dots, C^2)$ with $C = A + B$ or $B = A + C$ and $A, B, C > 0$ in strata $\mathcal{Q}(p-2, p-2, -2^p)$ with p odd. If among them, one quadratic residue is equal to 1 while another is equal to b_j^2, then the third value is the square of an element of $\mathbb{N} + \frac{1}{2}$. This would imply a existence of a fourth odd singularity. Therefore, the obstruction only impacts the case where there is no additional double pole. Consequently, we cannot reach a configuration forbidden by this obstruction by replacing an odd singularity by one with a bigger order.
From what precedes, it appears that if a distribution of angles is realized by a quadratic differential, then the singularities chosen to be in the equatorial net of the hemispherical surface (or equivalently the conical singularities of the flat metric) are the biggest possible. They realize the maximal integer sum.
Thus, we have to consider stratum $\mathcal{Q}(2a_1 - 2, \dots, 2a_{n_E} - 2, 2b_1 - 2, 2b_2 - 2, -2^t)$. We have $2t - 4 = 2\sum a_i - 2n_E + 2b_1 + 2b_2 - 4$. Consequently, $t = b_1 + b_2 + \sum a_i - n_E$. Among these double poles, one corresponds to b_3 and there are n_N non-integer singularities. The others are $K = t - 1 - n_N = b_1 + b_2 + \sum a_i - n_E - n_N - 1 = T + 2 - n$. The distribution of angles is realized by a hemispherical surface if and only if the configuration of quadratic residues formed by squares of numbers $c_1, \dots, c_{n_N}, b_3, 1, \dots, 1$ with K elements equal to 1 is realized in the stratum.
The first obstruction of Theorem 3.2 is already settled by eliminating distributions of angles described in Proposition 6.4. Since b_3 is not equal to any other number of the list, the second obstruction of Theorem 3.2 is not relevant. The third and the fourth obstruction are already encompassed in the arithmetic condition.
Finally, it should be noted that the obtained hemispherical surface has strict dihedral monodromy. Indeed, there are odd singularities on the equatorial net as well as on the poles of the latitude foliation. Therefore, the monodromy is not co-axial. □

Remark 6.6. In distribution of angles that are forbidden by the additional arithmetic condition, there should always be two non-integer singularities with equal angles (at least two elements among $r_1, \dots, r_{n_N+K+1}$ should be equal to one). They have smaller angle than any other singularity.

Example 6.7. Distribution of angles $3\pi, 3\pi, 3\pi, \frac{3\pi}{2}, \frac{3\pi}{2}$ cannot be realized by a metric with dihedral monodromy because of the additional arithmetic obstruction.

6.2.3. *Two odd singularities.* If $n_O = 2$, then it has already been proved in Section 4 of [3] that in absence of non-integer singularities, monodromy is co-axial. Therefore we will assume $n_N \geq 1$.

As previously, some specific obstructions have to be handled separately. They correspond to the first two obstructions of Theorem 3.2.

Proposition 6.8. *For $k \in \mathbb{N}$, $\alpha, \beta \notin \pi\mathbb{Z}$, the following distributions of angles are not realized by a cone spherical metric with strict dihedral monodromy on a punctured sphere:*

- $((2k+3)\pi, (2k+1)\pi, \alpha, \dots, \alpha, \beta, \beta)$ *with* $2k$ *angles equal to* α*;*
- $((2k+3)\pi, (2k+1)\pi, \alpha, \dots, \alpha)$ *with* $2k$ *angles equal to* α*;*
- $((2k+3)\pi, (2k+1)\pi, \alpha, \alpha)$*.*

Proof. If such a distribution is realized by a cone spherical metric, then it is also realized by a hemispherical surface. The latter should have at least two half-integer singularities on its equatorial net. This implies existence of a quadratic differential in $\mathcal{Q}(2k+1, 2k-1, -2^{2k+1})$ where zeroes are the two odd singularities of the spherical metric while the double poles correspond either to non-integer singularities or regular points. Therefore, quadratic residues can be equal to $\left(\frac{\alpha}{2\pi}\right)^2$ or $\left(\frac{\beta}{2\pi}\right)^2$ (if the double pole corresponds to a non-integer singularity) or are equal to one (if the double pole corresponds to a regular point of the spherical metric). In any case, the second obstruction of Theorem 3.2 forbids existence of a quadratic differential with such a configuration of quadratic residues. □

Proposition 6.9. *For $k \in \mathbb{N}$, $\alpha, \beta, \gamma \notin \pi\mathbb{Z}$, the following distributions of angles are not realized by a cone spherical metric with strict dihedral monodromy on a punctured sphere:*

- $((2k+3)\pi, (2k+3)\pi, \alpha, \dots, \alpha, \beta, \gamma)$ *with* $2k+1$ *angles equal to* α *and* $\alpha = \beta + \gamma$*;*
- $((2k+3)\pi, (2k+3)\pi, \alpha, \dots, \alpha, \beta, \alpha+\beta)$ *with* $2k+1$ *angles equal to* α*;*
- $((2k+3)\pi, (2k+3)\pi, \alpha, \dots, \alpha, \beta)$ *with* $2k+1$ *angles equal to* α*,* $\alpha + \beta = 2\pi$*,* $\beta = \alpha + 2\pi$ *or* $\alpha = \beta + 2\pi$*;*
- $((2k+3)\pi, (2k+3)\pi, \alpha, \alpha + 2\pi)$*;*
- $((2k+3)\pi, (2k+3)\pi, \alpha, \beta)$ *with* $\alpha + \beta = 2\pi$*.*

Proof. We proceed in the same way as in the proof of Proposition 6.8 except that in this case we refer to the first obstruction of Theorem 3.2. □

In addition to strengthened Gauss-Bonnet inequality (see Proposition 4.2) and obstructions of Proposition 6.8 and 6.9, an arithmetic condition should be satisfied if non-integer singularities have commensurable angles.

Theorem 6.10. *Let $2\pi(a_1, \dots, a_{n_E}, b_1, b_2, c_1, \dots, c_{n_N})$ be a distribution of angles with $n_N \geq 1$. We assume $b_1 \geq b_2$. Outside obstructions described in Proposition 6.8 and 6.9, there exists a cone spherical metric with strict dihedral monodromy on a punctured sphere with n conical singularities of prescribed angles if and only if the two following conditions hold:*

- *strengthened Gauss-Bonnet inequality* $T = \sum a_i + b_1 + b_2 \geq n - 2$ *holds;*
- *an additional arithmetic condition described below is satisfied when* $c_1, \dots, c_{n_N}$ *are commensurable.*

Considering vector $v = (c_1, \dots, c_{n_N}, 1, \dots, 1)$ with $T + 2 - n$ elements equal to 1, if v is of the form $L(r_1, \dots, r_{T-n_E})$ with $L > 0$ and $r_1, \dots, r_{T-n_E}$ integers, then:

- $\sum r_i \geq b_1 + b_2$*, if* $\sum r_i$ *is even;*
- $\sum r_i \geq b_1$*, if* $\sum r_i$ *is odd.*

Proof. We assume the distribution of angles satisfies strengthened Gauss-Bonnet inequality (Proposition 4.2) and is not concerned by obstructions of Proposition 6.8 and 6.9. We are going to prove existence of a cone spherical metric with strict dihedral monodromy in the

category of hemispherical surfaces.
Since strengthened Gauss-Bonnet inequality is satisfied, we can find in the distribution of angles a subset of s even or odd singularities of angles $\pi(2+d_i)$ such that $d_1+\cdots+d_s$ is an even number whose value is at least $4g-4+2(n-s)$. Among them there should be the two odd singularities. Then, we consider a nonempty stratum $\mathcal{Q}(d_1,\ldots,d_s,-2^t)$ of primitive quadratic differentials on the Riemann sphere.
Then, we have to check if the configuration of residues determined by the remaining singularities and the additional double poles is realized in the stratum. If one the four obstructions of Theorem 3.2 holds and one even singularity of angle $2a_i\pi$ is not among the s chosen singularities, then if we can just take stratum $\mathcal{Q}(2a_i-2,d_1,\ldots,d_n,-2^{t+a_i-1})$. The first two obstructions do not hold when there is zero of even order in differentials of the stratum. Concerning the two other obstructions, we add an integer residue. In any case, the sum of integer weights increases by at least one while the bound depends on the orders of the two odd singularities (in other words, the bound does not change). If the initial configuration is not realized in its stratum, it cannot be worse with the new one. Consequently, we assume every even singularity is among the s chosen singularities.

Thus, we have to consider stratum $\mathcal{Q}(2a_1-2,\ldots,2a_{n_E}-2,2b_1-2,2b_2-2,-2^t)$. We have $2t-4=2\sum a_i-2n_E+2b_1+2b_2-4$. Consequently, $t=b_1+b_2+\sum a_i-n_E$. Among these double poles, there are n_N non-integer singularities. The others are $K=t-n_N=b_1+b_2+\sum a_i-n_E-n_N=T+2-n$. The distribution of angles is realized by a hemispherical surface if and only if the configuration of quadratic residues formed by squares of numbers $c_1,\ldots,c_{n_N},1,\ldots,1$ with K elements equal to 1 is realized in the stratum.
The first and second obstruction are already settled by eliminating distributions of angles described in Propositions 6.8 and 6.9. The third and the fourth obstruction are already encompassed in the arithmetic condition.
Finally, the obtained hemispherical surface cannot have co-axial monodromy since it has odd singularities on the equator and at least one non-integer singularity at the poles since $n_N\geq 1$. □

6.3. **Comparison**. Just like in Proposition 5.4, some distributions of angles that can be realized by a spherical metric with co-axial monodromy can also be realized by a spherical metric with strict dihedral monodromy. We give a complete characterization.

Proposition 6.11. *Let $2\pi(a_1,\ldots,a_{n_E},b_1,\ldots,b_{n_O},c_1,\ldots,c_{n_N})$ be a distribution of angles. If it is realized by a spherical metric with co-axial monodromy on a punctured sphere, then it can also be realized by a metric with strict dihedral monodromy on a punctured sphere if and only if $n_O\geq 2$ and $n_O+n_N\geq 3$.*

Proof. Lemma 6.2 prove that $n_O\geq 2$ is a necessary condition for existence of a spherical metric with dihedral monodromy on a punctured sphere. Besides, if the number of singularities with nontrivial monodromy is exactly two, then the monodromy of the metric is automatically co-axial. Therefore, $n_O+n_N\geq 3$ is a necessary condition. We will prove that these two conditions are also necessary.
We consider a distribution of angles realized by a spherical metric with co-axial monodromy. We assume it satisfies the two necessary conditions. Theorem 6.1 implies in particular that $\sum a_i\geq n_E+n_O+n_N-2$. Since we have $n_O+n_N\geq 3$, we deduce $n_E\geq 1$. Therefore, the distribution of angles is not forbidden by Propositions 6.4, 6.8 and 6.9. Besides, the condition on the sum of orders of even singularities in Theorem 6.1 clearly implies the strengthened Gauss-Bonnet inequality (see Proposition 4.2). It remains to prove that the distribution of angles satisfies the hypothesis of Theorems 6.3, 6.5 and 6.10.
If $n_O\geq 4$, then there is no additional condition to check and Theorem 6.3 proves that the distribution of angles is realized by a spherical metric with strict dihedral monodromy.

If $n_O = 2$ or $n_O = 3$, we assume $b_1 \geq b_2 \geq b_3$. We already know that $\sum a_i \geq n-2$. We set $K = T + 2 - n$ and $T = \sum a_i + b_1 + b_2 \geq n-2$ (Strengthened Gauss-Bonnet inequality). Thus, $K \geq b_1 + b_2$. Theorem 6.5 and 6.10 require an additional arithmetic condition.
We first consider the case $n_O = 3$. The condition of Theorem 6.5 is the following. Let v be the vector $(c_1, \dots, c_{n_N}, b_3, 1, \dots, 1)$ with K elements equal to 1. If the distribution of angles cannot be realized by a metric with strict dihedral monodromy, then v is of the form $L(r_1, \dots, r_{n_N+K+1})$ with $L > 0$ and $r_1, \dots, r_{n_N+K+1}$ are integers. We should have:

- $\sum r_i \geq b_1 + b_2$ if $\sum r_i$ is even;
- $\sum r_i \geq b_1$ if $\sum r_i$ is odd.

If the condition is not satisfied, then $L = 1$ because otherwise the $K \geq b_1 + b_2$ elements of v that are equal to 1 would be enough to satisfy the bound. However, b_3 is not an integer, which leads to a contradiction since $\frac{b_3}{L}$ is required to be an integer.
In the case $n_O = 2$, the condition of Theorem 6.10 is essentially the same. We have to consider vector $(c_1, \dots, c_{n_N}, b_3, 1, \dots, 1)$ with K elements equal to 1. Similarly, if the condition is not satisfied, then $L = 1$. This implies $c_1, \dots, c_{n_N}$ are integers, which is a contradiction. We already know by hypothesis that $n_N \geq 1$. This ends the proof. □

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