

ON THE LOCUS OF GENUS 3 CURVES THAT ADMIT MEROMORPHIC DIFFERENTIALS WITH A ZERO OF ORDER 6 AND A POLE OF ORDER 2

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ABSTRACT. The main goal of this article is to compute the class of the divisor of $\overline{\mathcal{M}}_3$ obtained by taking the closure of the image of $\Omega\mathcal{M}_3(6; -2)$ by the forgetful map. This is done using Porteous formula and the theory of test curves. For this purpose, we study the locus of meromorphic differentials of the second kind, computing the dimension of the map of these loci to \mathcal{M}_g and solving some enumerative problems involving such differentials in low genus. A key tool of the proof is the compactification of strata recently introduced by Bainbridge-Chen-Gendron-Grushevsky-Möller.

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1. INTRODUCTION

The birational geometry of the moduli space \mathcal{M}_g of curves of genus g is an important topic in algebraic geometry. A way to understand it, is by studying the effective and ample divisors on \mathcal{M}_g . For example, Harris and Mumford [HM82, Har84] use the geometric properties of special classes of these divisors to compute, for genus $g \geq 23$, the Kodaira dimension of \mathcal{M}_g . Still many interesting questions related to the birational geometry of \mathcal{M}_g remain. To give an example, it is an open problem to know if the slope conjecture [HM90] holds asymptotically. In order to obtain new effective divisors in \mathcal{M}_g that help to understand more aspects of its geometry, we can take the projection of strata of abelian differentials to \mathcal{M}_g . The study of the divisors coming from holomorphic strata has been carried out in [Mul17], but the case of meromorphic strata is still completely open. The goal of this article is to give the first steps in this direction, that is, we present new techniques to study these divisors in the first interesting and non-trivial case.

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More precisely, let X be a smooth projective irreducible complex curve of genus g and K_X be the canonical line bundle on X . The global sections of K_X are the holomorphic differentials, and they form a vector space $H^0(X, K_X)$ of dimension g . A non-zero holomorphic differential ω over a curve X induces a translation structure on the complement of the zeroes of ω which can be realized as a plane polygon with certain side identifications by translations. Hence the pair (X, ω) is called indistinctly a *translation surface* or an *abelian differential* (see for example [MT02, Mö11]).

Let $\Omega\mathcal{M}_g$ be the moduli space of abelian differentials (X, ω) of genus g . This forms a vector bundle $\Omega\mathcal{M}_g \rightarrow \mathcal{M}_g$ whose fiber on $X \in \mathcal{M}_g$ is the space $H^0(X, K_X)$, which is called the *Hodge bundle*. Let $\mu = (a_1, \dots, a_n)$ a positive partition of $2g - 2$, that is, the integers $a_i \in \mathbb{Z}_{>0}$ satisfy $\sum_{j=1}^n a_j = 2g - 2$. The *stratum of abelian differentials* $\Omega\mathcal{M}_g(\mu)$ of *type* μ parametrises all couples $(X, \omega) \in \Omega\mathcal{M}_g$ with prescribed zeroes of order a_i at distinct points $z_i \in X$ for $i = 1, \dots, n$.

The above construction can be extended to the case of meromorphic abelian differentials. For every partition $\mu = (a_1, \dots, a_n; -b_1, \dots, -b_p)$ of $2g - 2$ with $a_i, b_j \geq 1$ there is a moduli space $\Omega\mathcal{M}_g(\mu)$ parametrising the pairs (X, ω) , where X is a genus g curve and ω a meromorphic differential with zeroes of order a_i at points z_i and poles of order b_j at points w_j . More details on the construction are given at the beginning of Section 2.

From the projection map $\Omega\mathcal{M}_g \rightarrow \mathcal{M}_g$ we have a projective bundle $\mathbb{P}(\Omega\mathcal{M}_g)$ over \mathcal{M}_g with fibre \mathbb{P}^{g-1} . The image of the stratum $\Omega\mathcal{M}_g(\mu)$ in $\mathbb{P}(\mathcal{M}_g)$ is called the *projective stratum* $\mathbb{P}\Omega\mathcal{M}_g(\mu)$. Similarly we can define the projective strata in the meromorphic case. In both cases, the projective strata $\mathbb{P}\Omega\mathcal{M}_g(\mu)$ parametrize abelian differentials modulo multiplication by non-zero complex scalars.

There is a natural map $\pi: \Omega\mathcal{M}_g(\mu) \rightarrow \mathcal{M}_g$ which factors through $\mathbb{P}\Omega\mathcal{M}_g(\mu)$ forgetting the differential. We denote by $\mathcal{M}_g(\mu)$ the image of the stratum $\Omega\mathcal{M}_g(\mu)$ in \mathcal{M}_g by π and by $\overline{\mathcal{M}}_g(\mu)$ its closure in the Deligne-Mumford moduli space of stable curves $\overline{\mathcal{M}}_g$. These loci are interesting subloci of $\overline{\mathcal{M}}_g$, but only a few results on them are known. The dimension of (the irreducible components of) $\mathcal{M}_g(\mu)$ have been computed in [Gen18] in the holomorphic case and [Bud20] in the meromorphic case. Moreover, in the holomorphic case, when the locus $\overline{\mathcal{M}}_g(\mu)$ is a divisor in $\overline{\mathcal{M}}_g$, its class has been computed in [Mul17].

In this paper, we study the loci $\overline{\mathcal{M}}_g(\mu)$ in the case of strata of meromorphic differentials. When these loci are divisors in $\overline{\mathcal{M}}_g$, we want to compute their class in the Picard group. In order to present our methods in a clear way, we treat as an example the stratum $\Omega\mathcal{M}_3(6; -2)$. Recall that the Picard group of $\overline{\mathcal{M}}_3$ is generated by the first Chern class λ of the Hodge bundle, and by the two boundary divisors δ_0 and δ_1 being the classes of the closures of the locus of irreducible singular curves and the locus obtained by gluing a smooth genus 1 curve to a smooth genus 2 curve respectively. Our main result is the following.

Theorem 1.1. *The class of $\overline{\mathcal{M}}_3(6; -2)$ in $\text{Pic}(\overline{\mathcal{M}}_3) \otimes \mathbb{Q}$ is*

$$(1) \quad [\overline{\mathcal{M}}_3(6; -2)] = 17108\lambda - 1792\delta_0 - 4396\delta_1.$$

The computation of the restriction of $[\overline{\mathcal{M}}_3(6; -2)]$ to $\text{Pic}(\mathcal{M}_3) \otimes \mathbb{Q} = \mathbb{Q} \cdot \lambda$ using Porteous formula is done in Section 4, thus giving the coefficient of λ in Equation (1). For the class in $\text{Pic}(\overline{\mathcal{M}}_3) \otimes \mathbb{Q}$ we use the theory of test curves and degeneration techniques in Section 5.

In order to perform the test curve in Section 5.3, some enumerative problems related to abelian differentials naturally appear. In particular, we consider the differentials with zero residues at all poles. These differentials are classically called *differentials of the second kind*. We denote by $\Omega\mathcal{R}_g(\mu)$ the locus of $\Omega\mathcal{M}_g(\mu)$ parametrizing the meromorphic differentials of type μ of the second kind. In Section 3 we compute the dimension of the image of $\Omega\mathcal{R}_g(\mu)$ by the forgetful map inside the moduli space of curves.

Theorem 1.2. *Let $\mu = (a_1, \dots, a_n; -b_1, \dots, -b_p)$ be a partition of $2g - 2$ such that $p \geq 2$ and $b_i \geq 2$ for all i .*

- *If $g = 1$ the dimension of the projection of every component of $\Omega\mathcal{R}_1(\mu)$ to $\mathcal{M}_{1,1}$ forgetting any subset of cardinal $n + p - 1$ of the singularities is 1.*
- *If $g \geq 2$, the dimension of the projection of $\Omega\mathcal{R}_g(\mu)$ to \mathcal{M}_g is $\min\{3g - 3; 2g + n - 2\}$.*

After giving some general results on families of stable curves and multi-scale differentials in Section 5.1, we solve some enumerative problems on differentials of the second kind in Section 5.2. The most interesting enumerative problem that we solve is the following one.

Theorem 1.3. *The map $\pi: \mathbb{P}\Omega\mathcal{R}_1(6; -2, -2, -2) \rightarrow \mathcal{M}_{1,1}$ forgetting the polar points is an unramified cover of degree 7.*

This result can be interpreted in the following way. On a fixed curve X of genus 1 there exist 7 differentials in $\Omega\mathcal{R}_1(6; -2, -2, -2)$ modulo translation on X and multiplication by \mathbb{C}^* of the differential.

To conclude, note that according to [Boi15] the strata of $\Omega\mathcal{M}_3(6; -2)$ has three connected components. Each component gives rise to an irreducible component of the divisor $\mathcal{M}_3(6; -2)$. We make some comments on this problem at the end of this paper (see Corollary 5.10) and in a future work we want to study each of these irreducible components.

Related works and possible applications. The study of the geometry of the strata of differentials recently attracted lots of interest. Sauvaget [Sau19] computes the Poincaré-dual cohomology classes of all strata in an inductive way. Extending the theory of the double ramification cycle, another group of authors computes in [BHP⁺20] the class of these strata in the tautological ring. Using this last result, our computation has been checked using sage package *admcycles* [DSvZ20] and can be similarly checked using the results of Sauvaget. On the other hand, our method gives some geometric information that seems difficult to obtain by their methods. Moreover, we believe that the generalisation of our result to other meromorphic strata can make more effective the computations of the software. Hence using both point of view on the strata, we should be able to deduce interesting properties of the birational geometry of \mathcal{M}_g .

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2. BACKGROUND

In this section we recall some known facts, first about the multi-scale differentials and then about Chern classes of the moduli space.

First we give some common background for both sections. Let consider the moduli space of smooth m -pointed genus g curves $\mathcal{M}_{g,m}$. Let $\pi: \mathcal{X} \rightarrow \mathcal{M}_{g,m}$ be the universal curve over $\mathcal{M}_{g,m}$, and let $\omega_\pi := \omega_{\mathcal{X}|\mathcal{M}_{g,m}}$ be the relative dualizing sheaf. We have that $\Omega\mathcal{M}_{g,m} := \pi_*(\omega_\pi)$ is a vector bundle over $\mathcal{M}_{g,m}$, that is the pull-back of the Hodge bundle.

Let n, p be strictly positive integers and we introduce the notation $m = n + p$. We denote by $\mu = (a_1, \dots, a_n; -b_1, \dots, -b_p)$ a m -tuple of integers such that $a_i, b_j \geq 1$ and $\sum_{j=1}^n a_j - \sum_{i=1}^p b_i = 2g - 2$. We study meromorphic differentials η with zeroes of order a_i at z_i and poles of order b_j at w_j , i.e. such that $(\eta) = (\eta)_0 - (\eta)_\infty = \sum a_j z_j - \sum b_i w_i$. We will usually write z for the tuple of points $(z_1, \dots, z_n; w_1, \dots, w_p)$. For $j = 1, \dots, m$, let \mathcal{D}_j be the sections of the universal curve corresponding to the marked points z_j if $j \leq n$ and w_{n+j} if $j \geq n + 1$. The strata $\Omega\mathcal{M}_g(\mu)$ of abelian differentials of type μ is defined to be the subspace of $\pi_*(\omega_\pi)(\sum_{i=1}^p b_i \mathcal{D}_{n+i})$ of differentials which vanish to order a_i at the sections \mathcal{D}_i up to the action of the group permuting the singularities of the same order.

2.1. The multi-scale differentials. In this section we recall some notions that we need about twisted and multi-scale differentials as introduced in [BCG⁺18] and [BCG⁺19].

We begin by recalling the notion of twisted differentials on which is based the more sophisticated notion of multi-scale differentials.

Definition 2.1. A *twisted differential* ω of type μ (compatible with a full order \leq) on a stable n -pointed curve (X, z) is a collection of (possibly meromorphic) differentials ω_v on the irreducible components X_v of X and a full order \leq on the set of these components, such that no ω_v is identically zero, with the following properties:

- (0) **(Vanishing as prescribed)** Each differential ω_v is holomorphic and non-zero outside of the nodes and marked points of X_v . Moreover, if a marked point z_j lies on X_v , then $\text{ord}_{z_j} \omega_v = m_j$.
- (1) **(Matching orders)** For any node of X that identifies $q_1 \in X_{v_1}$ with $q_2 \in X_{v_2}$,

$$\text{ord}_{q_1} \omega_{v_1} + \text{ord}_{q_2} \omega_{v_2} = -2.$$

- (2) **(Matching residues at simple poles)** If at a node of X that identifies $q_1 \in X_{v_1}$ with $q_2 \in X_{v_2}$ the condition $\text{ord}_{q_1} \omega_{v_1} = \text{ord}_{q_2} \omega_{v_2} = -1$ holds, then $\text{Res}_{q_1} \omega_{v_1} + \text{Res}_{q_2} \omega_{v_2} = 0$.
- (3) **(Partial order)** If a node of X identifies $q_1 \in X_{v_1}$ with $q_2 \in X_{v_2}$, then $v_1 \succcurlyeq v_2$ if and only if $\text{ord}_{q_1} \omega_{v_1} \geq -1$. Moreover, $v_1 \asymp v_2$ if and only if $\text{ord}_{q_1} \omega_{v_1} = -1$.

- (4) **(Global residue condition)** For every level i and every connected component Y of $X_{>i}$ that does not contain a pole w_i the following condition holds. Let q_1, \dots, q_b denote the set of all nodes where Y intersects $X_{(i)}$. Then

$$\sum_{j=1}^b \text{Res}_{q_j^-} \omega = 0,$$

where by definition $q_j^- \in X_{(i)}$.

Note that by point (1) of Definition 2.1, at each node of X the twisted differential ω , either has two simple poles or has a zero of order k on one branch of the node and a pole of order $-k - 2$ on the other branch. The *prong number* at a node is 0 in the first case and $\kappa = k + 1$ in the second case.

Now we give the definition of a multi-scale differential, referring to [BCG⁺19] for details.

Definition 2.2. A *multi-scale differential* $(X, z, \omega, \leq, \sigma)$ of type μ is a stable pointed curve (X, z) with $z = (z_1, \dots, z_n; w_1, \dots, w_p)$, a twisted differential ω of type μ over X compatible with the total order \leq and a global prong-matching σ .

The notion of prong-matching is introduced and discussed in great details in [BCG⁺19]. For us it will not be crucial to know its precise definition. It is sufficient to now that it gives a way to glue the differentials at the nodes. We will use mainly the facts that it is a finite data and that the number of possible classes of prong-matching is computable. In the important case of a multi-scale differential with two levels, this number is

$$(2) \quad \kappa = \text{gcd}(\kappa_i),$$

where i runs through the set of nodes of the multi-scale differential.

The importance of the notion of multi-scale differentials comes from the following theorem proved in [BCG⁺19].

Theorem 2.3. *The stratum $\Omega\mathcal{M}_g(\mu)$ is an open dense subset of the moduli space $\Xi\mathcal{M}_g(\mu)$ of the multi-scale differentials of type μ . Moreover, the projectivisation of $\Xi\mathcal{M}_g(\mu)$ is a compactification of the projectivized stratum $\mathbb{P}\Omega\mathcal{M}_g(\mu)$.*

Moreover, there exists a good system of coordinates near the boundary of this space (see [BCG⁺19, Section 11]). The perturbed periods coordinates give a way to understand the families of degenerating differentials with special properties. In this article, the relevant information that we need is that near a given multi-scale differential, the top differential is a small deformation away from the nodal points, while on the lower levels the differentials are multiplied by $\prod t_i^{a_i}$, where t_i is a local parameter for each level of the multi-scale differential and a_i is an integer defined in [BCG⁺19, Equation (6.7)]. Moreover, we know the local equations of the nodes in the universal family.

In the useful case of a multi-scale differential with two levels we can be more specific. The local equation of the family at the node n_i is $x_i y_i = t^{a_i}$ for a local parameter t and with $a_i = \text{lcm}(\kappa_n)/\kappa_i$.

2.2. Chern classes. The goal of this section is to recall some facts about Chern classes on the moduli space of curves. In order to make this section self contained, we begin by recalling some well-know facts of algebraic geometry.

2.2.1. Some Notation. Let $f: Z \hookrightarrow Y$ be a closed immersion of schemes, and denote the sheaf of Kähler differentials of Z over Y by $\Omega_{Z|Y}^1$. We have the exact sequence

$$0 \rightarrow I_Z \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_Z \rightarrow 0,$$

where $I = I_Z$ is the ideal sheaf of Z and $\mathcal{O}_Y/I_Z = \mathcal{O}_Z$. Let F be a locally free sheaf on Y , then tensoring the above exact sequence by F we get

$$0 \rightarrow I_Z \otimes F \rightarrow F \rightarrow F_Z \rightarrow 0,$$

where $F_Z = F \otimes \mathcal{O}_Z$. The conormal sheaf $\mathcal{N}_{Z|Y}^\vee$ of f is the quasi-coherent \mathcal{O}_Z -module I/I^2 and the normal sheaf is $\mathcal{N}_{Z|Y} = \text{Hom}_{\mathcal{O}_Z}(I/I^2, \mathcal{O}_Z)$. Suppose now that Z is an effective Cartier divisor and let $F = \mathcal{O}_Y(Z)$ the associated invertible sheaf, then we have an exact sequence

$$0 \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_Y(Z) \rightarrow \mathcal{O}_Z(Z) \rightarrow 0,$$

where $\mathcal{O}_Z(Z) = \mathcal{O}_Y(Z)|_Z$. In this case we have that $\mathcal{N}_{Z|Y}^\vee = \mathcal{O}_Y(-Z)|_Z$ and $\mathcal{N}_{Z|Y} = \mathcal{O}_Y(Z)|_Z$. From the exact sequence

$$0 \rightarrow \mathcal{O}_Y(-2Z) \rightarrow \mathcal{O}_Y(-Z) \rightarrow \mathcal{O}_Y(-Z)|_Z \rightarrow 0$$

we have that

$$\Omega_{Z|Y}^1 \simeq I/I^2 = \mathcal{O}_Y(-Z)/\mathcal{O}_Y(-2Z) = \mathcal{O}_Y(-Z)|_Z = \mathcal{N}_{Z|Y}^\vee$$

and

$$(3) \quad (\Omega_{Z|Y}^1)^\vee \simeq \mathcal{O}_Y(Z)/\mathcal{O}_Y = \mathcal{O}_Y(Z)|_Z = \mathcal{N}_{Z|Y}.$$

We recall that if A is a ring and $N \subset M \subset L$ are A -modules, then $L/M \simeq (L/N)/(M/N)$. In our case we have the inclusion $\mathcal{O}_Y \subset \mathcal{O}_Y((n-1)Z) \subset \mathcal{O}_Y(nZ)$ of coherent sheaves on Y . Hence we consider for every $n \geq 1$ the quotient sheaves to get the following exact sequences on Y

$$(4) \quad 0 \rightarrow \mathcal{O}_Y((n-1)Z)/\mathcal{O}_Y \rightarrow \mathcal{O}_Y(nZ)/\mathcal{O}_Y \rightarrow \mathcal{O}_Y(nZ)/\mathcal{O}_Y((n-1)Z) \rightarrow 0.$$

2.2.2. The setting. Given any family $\pi: \mathcal{X} \rightarrow B$ of curves of genus g , we denote by $\omega_{\mathcal{X}|B}$ the relative dualizing sheaf of the family π . When the family π contains singular fibers, we have that $\omega_{\mathcal{X}|B}$ is equal to $\Omega_{\mathcal{X}|B}^1$ away of the nodes of the fibers, thus, when the family is a family of smooth curves we can identify $\omega_{\mathcal{X}|B} \simeq \Omega_{\mathcal{X}|B}^1$.

Consider the universal curve $\pi_0: \mathcal{X} = \mathcal{M}_{g,1} \rightarrow \mathcal{M}_g$. For $g \geq 2$ this map is smooth of relative dimension one. We denote by Ω the relative dualizing sheaf $\omega_{\mathcal{X}|\mathcal{M}_g}$ associated to π_0 . Let $\pi^n: \mathcal{X}^n \rightarrow \mathcal{M}_g$ be the n -fold fiber product of \mathcal{X} over \mathcal{M}_g . The space \mathcal{X}^n parametrises smooth genus g curves with n -tuples of not necessary distinct points. Note that the fiber over $X \in \mathcal{M}_g$ of π^n is the direct product $X^n = X \times \cdots \times X$ and the fiber of $\mathcal{M}_{g,n} \rightarrow \mathcal{M}_g$ is the complement of the diagonal Δ in X^n , where

$$\Delta = \{(X, q_1; \dots, q_n) : q_i = q_j \text{ for at least two indices } i \neq j\}.$$

Let Δ_{ij} be the diagonal corresponding the points where $q_i = q_j$ for two indices $i \neq j$. Let $\pi_i: \mathcal{X}^n \rightarrow \mathcal{X}$ be the forgetful map which forgets all but the i -th factor and let define the sheaf $\Omega_i = \pi_i^*(\Omega)$.

2.2.3. *The Chern classes for $\pi_*(\Omega_2(2\Delta))$.* We restrict our attention to the case $n = 1$, that is, we consider the projection $\pi = \pi_1: \mathcal{X}^2 \rightarrow \mathcal{X}$ on the first factor. We will write $\mathcal{O} = \mathcal{O}_{\mathcal{X}^2}$ and consider the diagonal of the fiber product on \mathcal{X}^2 , that is, $\Delta: \mathcal{X} \rightarrow \mathcal{X}^2 = \mathcal{X} \times_{\mathcal{M}_g} \mathcal{X}$, and we identify $\Delta(\mathcal{X}) \simeq \mathcal{X}$. We write Δ to denote the image $\Delta(\mathcal{X})$. From identifications in Equation (3) and from the exact sequence (4) with $Z = \Delta$ and $Y = \mathcal{X}^2$, we have that the normal bundle $\mathcal{N}_{\Delta|\mathcal{X}^2}$ satisfies

$$(5) \quad \mathcal{N}_{\Delta|\mathcal{X}^2} = \mathcal{O}_{\mathcal{X}^2}(\Delta)/\mathcal{O}_{\mathcal{X}^2} \simeq (\Omega_{\Delta|\mathcal{X}^2}^1)^\vee \simeq (\pi_2^*(\Omega_{\mathcal{X}|\mathcal{M}_g}^1))^\vee = (\Omega_2)^\vee = \pi_2^*(\Omega)^\vee.$$

With this notation, we twist the sequence (4) for $n = 2$ by $\Omega_2 = \pi_2^*(\Omega)$ to get the exact sequence

$$0 \rightarrow \Omega_2 \otimes (\Omega_{\Delta|\mathcal{X}^2}^1)^\vee \rightarrow \Omega_2 \otimes (\mathcal{O}(2\Delta)/\mathcal{O}) \rightarrow \Omega_2 \otimes ((\Omega_{\Delta|\mathcal{X}^2}^1)^2)^\vee \rightarrow 0.$$

Using Equation (5) the previous exact sequence reads

$$0 \rightarrow \mathcal{O}_{\mathcal{X}^2} \rightarrow \Omega_2 \otimes (\mathcal{O}(2\Delta)/\mathcal{O}) \rightarrow (\Omega_2)^\vee \rightarrow 0.$$

Pushing down this exact sequence to \mathcal{X} with π_* we have the following exact sequence of coherent sheaves

$$(6) \quad 0 \rightarrow F_0 \rightarrow F_1 \rightarrow F_2 \rightarrow F_3 \rightarrow 0,$$

where $F_0 := \pi_*(\mathcal{O}_{\mathcal{X}^2}) = \mathcal{O}_{\mathcal{X}}$, $F_1 := \pi_*(\Omega_2 \otimes (\mathcal{O}(2\Delta)/\mathcal{O}))$, $F_2 := \pi_*(\Omega_2^\vee)$ and $F_3 := R^1\pi_*\mathcal{O}_{\mathcal{X}^2} = \mathcal{O}_{\mathcal{X}}$. The equality $R^1\pi_*(\Omega_2 \otimes (\mathcal{O}(2\Delta)/\mathcal{O})) = 0$ follows from the fact that $\Omega_2 \otimes (\mathcal{O}((b-1)\Delta)/\mathcal{O})$ is supported on Δ (since the latter is isomorphic to $\Omega_2(2\Delta)|_{2\Delta}$). The fiber of F_1 at a point $(X, p) \in \mathcal{X}$ is the two-dimensional vector space of sections $H^0(X, K_X(2p)/K_X)$. Similarly the fibers of F_0 and F_3 at (X, p, q) are $H^0(X, \mathcal{O}_X)$ and the fiber of F_2 is $H^0(X, K_X^\vee) = \{0\}$. So the exact sequence (6) reads

$$0 \rightarrow \mathcal{O}_{\mathcal{X}} \rightarrow F_1 \rightarrow \pi_*(\Omega_2^\vee) \rightarrow \mathcal{O}_{\mathcal{X}} \rightarrow 0.$$

We split this sequence into the two short exact sequences

$$0 \rightarrow \mathcal{O}_{\mathcal{X}} \rightarrow F_1 \rightarrow \text{Im} \rightarrow 0$$

and

$$0 \rightarrow \text{Coker} \rightarrow \mathcal{O}_{\mathcal{X}} \rightarrow 0,$$

where Im and Coker are the image and the cokernel of the map $F_1 \rightarrow \pi_*(\Omega_2^\vee)$. The fact that $c(\mathcal{O}_{\mathcal{X}}) = 1$ and the additivity of Chern classes imply that $c(\text{Im}) = c(\pi_*(\Omega_2^\vee))$. Hence the Chern classes satisfy

$$(7) \quad c(F_1) = c(\mathcal{O}_{\mathcal{X}^2})c(\pi_*(\Omega_2^\vee)) = 1 - K_1,$$

where, following the notation in [Fab99], we have that $K_1 = c_1(\pi_*(\Omega_2^\vee))$ is the first Chern class.

Let $E = \pi_*(\Omega_2)$ and $E_2 = \pi_*(\Omega_2 \otimes \mathcal{O}_{\mathcal{X}^2}(2\Delta)) = \pi_*(\Omega_2(2\Delta))$. Since $R^1\pi_*\Omega_2 = \mathcal{O}_{\mathcal{X}}$, we have the exact sequence

$$0 \rightarrow E \rightarrow E_2 \rightarrow F_1 \rightarrow \mathcal{O}_{\mathcal{X}} \rightarrow 0.$$

Using the same trick as previously this implies that the Chern classes satisfy the equality $c(E_2) = c(E)c(F_1) = (1 + \lambda)(1 - K_1)$. This leads to the following result.

Proposition 2.4. *The Chern class of E_2 is*

$$c(E_2) = 1 + \sum_{i \geq 1} (\lambda_i - \lambda_{i-1} K_1).$$

To conclude, let us remark that we can define a similar vector bundle on \mathcal{X}^n . It suffices to consider the exact sequence

$$0 \rightarrow \mathcal{O}_{\mathcal{X}^n} \rightarrow \Omega_2 \otimes (\mathcal{O}(2\Delta_{n,n+1})/\mathcal{O}) \rightarrow (\Omega_2)^\vee \rightarrow 0$$

and to define E_2 to be the push-forward by the map forgetting the last point of the middle term. Then Proposition 2.4 remains true in this generalised context.

2.2.4. The Chern classes for $\pi_*(\Omega_2(b\Delta))$ with $b \geq 3$. Now we want to extend the result of the formula of the Chern class of Proposition 2.4 to the bundle $E_b := \pi_*(\Omega_2(b\Delta))$ on \mathcal{X} for all $b \geq 3$. Note that the fiber of E_b at a point $(X, p) \in \mathcal{X}$ is the space $H^0(X, K_X(bp))$ of differentials on X that have at worst poles of order b at p .

As in the previous subsection $R^1\pi_*(\Omega_2 \otimes \mathcal{O}((b-1)\Delta)/\mathcal{O}) = 0$ since the sheaf is supported on the diagonal. Using this fact, twisting the exact sequence (4) by Ω_2 and using Equation (5) we obtain the exact sequence

$$0 \rightarrow \pi_*(\Omega_2 \otimes \mathcal{O}((b-1)\Delta)/\mathcal{O}) \rightarrow \pi_*(\Omega_2 \otimes \mathcal{O}(b\Delta)/\mathcal{O}) \rightarrow \pi_*((\Omega_2^{\otimes b-1})^\vee) \rightarrow 0$$

on \mathcal{X} . This gives a recursive formula for Chern classes of E_b as follow

$$c(\pi_*(\Omega_2 \otimes \mathcal{O}(b\Delta)/\mathcal{O})) = c(\pi_*(\Omega_2 \otimes \mathcal{O}((b-1)\Delta)/\mathcal{O}))c(\pi_*((\Omega_2^{\otimes b-1})^\vee)).$$

Since the base case $c(\pi_*(\Omega_2 \otimes \mathcal{O}(2\Delta)/\mathcal{O})) = (1 - K_1)$ is given by equation (7), we obtain

$$(8) \quad c(\pi_*(\Omega_2 \otimes \mathcal{O}(b\Delta)/\mathcal{O})) = \prod_{i=1}^{b-1} (1 - iK_1).$$

Let X be a smooth curve and $p \in X$. For all positive integer b we have the following exact sequence

$$0 \rightarrow K_X \rightarrow K_X(bp) \rightarrow K_X(bp)|_{bp} \rightarrow 0.$$

In order to globalize this exact sequence to the universal curve \mathcal{X} we proceed as follow. On \mathcal{X}^2 we have the following exact sequence

$$(9) \quad 0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(b\Delta) \rightarrow (\mathcal{O}(b\Delta))|_{b\Delta} \rightarrow 0.$$

Tensoring with Ω_2 and pushing down to \mathcal{X} we obtain, using the fact that $R^1\pi_*\Omega_2 = \mathcal{O}_{\mathcal{X}}$, the following exact sequence

$$0 \rightarrow \pi_*(\Omega_2) \rightarrow \pi_*(\Omega_2(b\Delta)) \rightarrow \pi_*(\Omega_2(b\Delta)|_{b\Delta}) \rightarrow \mathcal{O}_{\mathcal{X}} \rightarrow 0.$$

Since the exact sequence (9) gives that $\mathcal{O}_{\mathcal{X}^2}(b\Delta)/\mathcal{O}_{\mathcal{X}^2} = (\mathcal{O}(b\Delta))|_{b\Delta}$, we identify the vector bundle $\pi_*(\Omega_2(b\Delta)|_{b\Delta})$ with $\pi_*(\Omega_2 \otimes (\mathcal{O}(b\Delta)/\mathcal{O}))$. Hence using the same trick as in the case $b = 2$, Equation (8) gives the class $c(E_b)$.

Proposition 2.5. *The Chern class of E_b is*

$$c(E_b) = (1 + \lambda) \prod_{i=1}^{b-1} (1 - iK_1).$$

Finally note that, as in the case of E_2 , this proposition can be extended on \mathcal{X}^n to the sheaf E_b similarly defined.

2.2.5. Known facts about Chern Classes. We conclude this section by recalling two known facts about Chern classes. The first one is the inversion formula for Chern classes. The second is some equalities for the Chern classes above the moduli space.

We first give a formula in order to compute the inverse of a Chern class. Much more material around this circle of ideas can be found in [Hir56]. Let us first define the polynomial

$$P_n(x_1, \dots, x_n) = \sum_{i_1+2i_2+\dots+ni_n=n} \left(\frac{(i_1 + \dots + i_n)!}{i_1! \dots i_n!} \prod_{j=1}^n (-x_j)^{i_j} \right).$$

To be concrete, the polynomials P_n for $n \leq 3$ are

$$\begin{aligned} P_0 &= 1, \\ P_1 &= -x_1, \\ P_2 &= x_1^2 - x_2, \\ P_3 &= -x_1^3 + 2x_1x_2 - x_3. \end{aligned}$$

The importance of these polynomials is given by the following result that will be used several times in Section 5.

Lemma 2.6. *Let E be a complex vector bundle over a complex manifold X whose Chern class is $c(E) = 1 + c_1(E) + c_2(E) + \dots + c_n(E)$. Then the Chern class of the dual E^\vee of E is*

$$c(E^\vee) = 1 + P_1(c_1) + P_2(c_1, c_2) + \dots + P_n(c_1, \dots, c_n).$$

Note that the Chern class of E^\vee is also known as the (total) Segre class of E . Moreover we will usually denote E^\vee by $-E$, viewing it as an element of the Grothendieck group (see [HM98, Section 3.E]).

We now recall some results about Chern classes on the moduli space. These results are due to [HM82] and many examples of application can be find in [Fab99].

Lemma 2.7. *Let K_i be the class of $\Omega_i = \pi_i^*(\Omega)$ and Δ_{ij} be (the class of) the (i, j) -th diagonal as introduced in Section 2.2.2, then*

$$\begin{aligned} \Delta_{id}\Delta_{jd} &= \Delta_{ij}\Delta_{id} \text{ for } i < j < d, \\ \Delta_{ij}^2 &= -K_i\Delta_{ij} \text{ for } i < j, \\ K_j\Delta_{ij} &= K_i\Delta_{ij} \text{ for } i < j. \end{aligned}$$

Moreover for every monomial M pulled back from \mathcal{X}^{d-1} we have

$$\begin{aligned} \pi_{d,*}(M \cdot \Delta_{id}) &= M, \\ \pi_{d,*}(M \cdot K_d^k) &= M \cdot p^*(\kappa_{k-1}), \end{aligned}$$

where $p: \mathcal{X}^{d-1} \rightarrow \mathcal{M}_g$ is the map forgetting all the marked points.

3. THE LOCUS OF DIFFERENTIALS OF THE SECOND KIND

The objective of this section is to compute the dimension of the projection to \mathcal{M}_g of the locus parametrizing the differentials of second kind in the strata of meromorphic differentials. Recall that the differentials of second kind are meromorphic differentials such that the residue at every pole is zero. We denote the locus of differentials of type $\mu = (a_1, \dots, a_n; -b_1, \dots, -b_p)$ of second kind of type μ by $\Omega\mathcal{R}_g(\mu)$.

We begin with a preliminary result.

Lemma 3.1. *Given a stratum of meromorphic differentials $\Omega\mathcal{M}_g(a_1, \dots, a_n; -b_1, \dots, -b_p)$ such that $g \geq 1$, $p \geq 2$ and $b_i \neq 1$. The subspace which parametrises differentials with zero residues at the first $i \leq p - 1$ points is of codimension i .*

Proof. By [GT21] these subspaces are not empty. Moreover, each condition on the residues gives a linear equation in period coordinates. Since these equations involve at most $p - 1$ residues, the i equations are independent. Hence the locus that they define is of codimension i . \square

We can now compute the dimension of the projection of the loci of differentials of second kind to \mathcal{M}_g , proving Theorem 1.2. Recall that this theorem says that the dimension of the projection is 1 in the genus one case and $\min\{3g - 3; 2g + n - 2\}$ in the genus $g \geq 2$ case. The proof is by degeneration in the spirit of Theorem 1.3 of [Gen18]. In this proof, we use the notation $b = \sum_{i=1}^p b_i$.

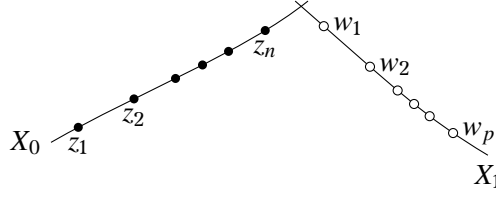
Proof. We begin with the case $g = 1$. In that case, Lemma 3.1 implies that the dimension of the space $\Omega\mathcal{R}_g(\mu)$ is equal to $n + 1$. If there were a component of $\Omega\mathcal{R}_1(\mu)$ such that the dimension of the projection is zero, then there exists a curve X having a $n + 1$ dimensional family of differentials of the second kind of type μ on X . We know (see for example Section 2 of [Hur00]) that a differential of type μ on X can be written

$$(10) \quad \omega = \lambda \frac{\sigma^{a_1}(z - z_1) \cdots \sigma^{a_n}(z - z_n)}{\sigma^{b_1}(z - w_1) \cdots \sigma^{b_p}(z - w_p)} dz,$$

where σ is Weierstraß sigma-function of X , the sum $\sum a_i z_i - \sum b_j w_j = 0$ and $\lambda \in \mathbb{C}^*$. Without lose of generality we suppose that the forgetful map to $\mathcal{M}_{1,1}$ keep the point z_1 . Hence the dimension of the space of differentials given by Equation (10) on the elliptic curve (X, z_1) is $n + p - 1$. It is easy to check that the residue map given the residues of ω at w_1, \dots, w_{p-1} is a non constant rational function from \mathbb{C}^{n+p-1} to \mathbb{C}^{p-1} . Hence the variety given by this equation is of dimension n and can not be a whole component of the locus $\Omega\mathcal{R}_g(\mu)$.

For $g \geq 2$, we degenerate to the curve X pictured in Figure 1. The curve X_0 is of genus $g - 1$ and the genus of X_1 is 1. We denote by p_0 and p_1 the nodal points belonging respectively to X_0 and X_1 .

We consider the twisted differential ω on X such that the restriction ω_i to the irreducible component X_i are the following differentials. On X_0 the differential is in the stratum $\Omega\mathcal{M}_{g-1}(a_1, \dots, a_n; -b - 2)$ and on X_1 the differential is in $\Omega\mathcal{R}_1(b; -b_1, \dots, -b_p)$. Note that by [Mul19, Theorem 1.1] this twisted differential is in the closure of $\Omega\mathcal{R}_g(\mu)$. Moreover by [Bud20] the dimension of the projection $\Omega\mathcal{M}_{g-1}(a_1, \dots, a_n; -b - 2)$ to \mathcal{M}_{g-1} is $\min(3(g - 1) - 3, 2(g - 1) - 2 + n)$.


 FIGURE 1. The pointed curve X we are degenerated to.

Suppose that $n \geq g - 1$, then there exist a dense subset U of $\overline{\mathcal{M}}_{g-1}$ such that for every point X'_0 in U there exists a differential ω_0 on X'_0 . Moreover there is a positive dimensional family of such differentials with the polar point p_0 moving on X'_0 . By the case of genus $g = 1$, the dimension of the projection of $\Omega\mathcal{R}_1(b; -b_1, \dots, -b_p)$ is equal to one. Moreover, there is one smoothing parameter at the node. Summing up the contribution, the dimension of the projection is $3(g - 1) - 3 + 1 + 1 + 1 = 3g - 3$, where $3(g - 1) - 3$ is the dimension of the projection of $\Omega\mathcal{M}_{g-1}(a_1, \dots, a_n; -b - 2)$ to \mathcal{M}_{g-1} and the three 1 are respectively the moving point $p_0 \in X'_0$, the dimension of the space of elliptic curves $\{(X_1, p_1)\}$ and the smoothing parameter of the node.

If $n < g - 1$ then by [Bud20] there exists a $2(g - 1) - 2 + n$ dimension space of curves X_0 which admits such differential ω_0 . Moreover the set of possible point p_0 is finite on X_0 . Hence the dimension of the projection of the locus $\Omega\mathcal{R}_g(\mu)$ is $2(g - 1) - 2 + n + 1 + 1 = 2g - 2 + n$ in this case. In this sum, the contribution $2(g - 1) - 2 + n$ is the dimension of the projection of $\Omega\mathcal{M}_{g-1}(a_1, \dots, a_n; -b - 2)$ to \mathcal{M}_{g-1} and the remaining contribution from the dimension of the space of elliptic curves $\{(X_1, p_1)\}$ and the smoothing parameter of the node. \square

4. THE CLASS OF THE DIVISOR $\mathcal{M}_3(6; -2)$

The goal of this section is to compute the class of the projection $\mathcal{M}_3(6; -2)$ of $\Omega\mathcal{M}_3(6; -2)$ to \mathcal{M}_3 . This gives the coefficient of the λ -class in Equation (1) of Theorem 1.1. In order to do this, we use Porteus Formula and the results recalled in Section 2.2.

For $n \geq 1$, set $\mathcal{O} = \mathcal{O}_{\mathcal{X}^n}$. Recall that the diagonal Δ_{ij} is given by the locus where $q_i = q_j$ in \mathcal{X}^3 and define the divisor $\mathfrak{D}_2^6 = 2\Delta_{23} - 6\Delta_{13}$ inside \mathcal{X}^3 . Tensoring the exact sequence

$$0 \rightarrow \mathcal{O}(-6\Delta_{13}) \rightarrow \mathcal{O} \rightarrow \mathcal{O}|_{6\Delta_{13}} \rightarrow 0$$

by $\Omega_3(2\Delta_{23})$ we obtain

$$0 \rightarrow \Omega_3 \otimes \mathcal{O}(-6\Delta_{13} + 2\Delta_{23}) \rightarrow \Omega_3 \otimes \mathcal{O}(2\Delta_{23}) \rightarrow \Omega_3 \otimes \mathcal{O}(2\Delta_{23})|_{6\Delta_{13}} \rightarrow 0.$$

Pushing down this exact sequence on \mathcal{X}^2 by the map π forgetting the third point, we obtain

$$0 \rightarrow \pi_*(\Omega_3(\mathfrak{D}_2^6)) \rightarrow \pi_*(\Omega_3 \otimes \mathcal{O}(2\Delta_{23})) \rightarrow \pi_*(\Omega_3(2\Delta_{23})|_{6\Delta_{13}}) \rightarrow 0.$$

We define the following coherent sheaf on \mathcal{X}^2

$$\mathcal{F} = \pi_*(\Omega_3 \otimes (\mathcal{O}(2\Delta_{23})/(\mathcal{O}(\mathfrak{D}_2^6)))).$$

Note that the stalk \mathcal{F}_x of \mathcal{F} at a point $x = (X; z, w) \in \mathcal{X}^2$ is given by

$$\mathcal{F}_x = H^0(X, K_X(2w)/K_X(2w - 6z)).$$

The evaluation map gives a morphism $\phi: E_2 \rightarrow \mathcal{F}$ over \mathcal{X}^2 such that on the stalk we have

$$\phi_x: (E_2)_x = H^0(X, K_X(2w)) \rightarrow \mathcal{F}_x = H^0(X, K_X(2w)/K_X(2w - 6z)).$$

Since the dimension of the source is 4 and the dimension of the target is 6, by Porteous formula (see [HM98, Theorem 3.114]) the class of the degeneracy locus where the map has rank less or equal to three is $c_3(\mathcal{F} - E_2)$. Note that this degeneracy locus can contain extra components in the diagonal $\Delta_{1,2} = \{(w, z) : w = z\}$. This is indeed the case and we will deal with this problem at the end of this section.

The Chern classes of \mathcal{F} is given by the following general formula.

Lemma 4.1. *The Chern class of the vector bundle whose fiber is*

$$H^0\left(K\left(\sum_{i=1}^p b_i w_i\right)/K\left(\sum_{i=1}^p b_i w_i - \sum_{i=1}^n a_i z_i\right)\right)$$

is equal to

$$\prod_{i=1}^n \prod_{j=1}^{a_i} (1 + jK_{p+i} + \sum_{k=1}^p b_k \Delta_{k,p+i}).$$

Proof. Given an effective divisor D on a smooth curve X , the sheaf $K_X(D)$ is the sheaf of meromorphic 1-forms ω such that for a point q on X , $(\operatorname{div} \omega)(q) + D(q) \geq 0$, in particular if $q \notin \operatorname{supp}(D)$, then ω is holomorphic at q . If q is a zero of order a for ω , then in an open neighbourhood U around q with local coordinate z , we can trivialize K_X so that the form $z^a \cdot \omega$ generates the stalk $K_X(D)|_q$, then $K_X(D)|_q \simeq K_X^{\otimes a}|_q \otimes \mathcal{O}_q(D)$.

Recall that we denote by $\mu = (a_1, \dots, a_n; -b_1, \dots, -b_p)$ a partition of $2g - 2$ where $a_i, b_j \geq 1$ and that $m = n + p$.

On \mathcal{X}^m we denote $\Delta_Z = \sum_{i=1}^n a_i \Delta_{p+i, m+1}$ and $\Delta_W = \sum_{j=1}^p b_j \Delta_{j, m+1}$ the divisor of zeros and poles respectively. We then consider the divisor $\Delta_{Z,W} = \Delta_Z - \Delta_W$ and the following two exact sequences on \mathcal{X}^{m+1} and \mathcal{X}^m respectively:

$$0 \rightarrow \mathcal{O}(\Delta_W - \Delta_Z) \rightarrow \mathcal{O}(\Delta_W) \rightarrow \mathcal{O}(\Delta_W)|_{\Delta_Z} \rightarrow 0,$$

$$0 \rightarrow \pi_*(\Omega_{m+1}(\mathcal{O}(\Delta_W - \Delta_Z))) \rightarrow \pi_*(\Omega_{m+1}(\mathcal{O}(\Delta_W))) \rightarrow \pi_*(\Omega_{m+1}(\mathcal{O}(\Delta_W)|_{\Delta_Z})) \rightarrow 0.$$

We recall that $\mathcal{O}(\Delta_W)|_{\Delta_Z} = \mathcal{O}_{\Delta_Z}(\Delta_W) = \mathcal{O}(\Delta_W) \otimes \mathcal{O}_{\Delta_Z}$. We set

$$\mathcal{F} := \pi_*(\Omega_{m+1}(\mathcal{O}(\Delta_W)|_{\Delta_Z})) = \pi_*(\Omega_{m+1}(\mathcal{O}(\Delta_W))/\mathcal{O}(\Delta_W - \Delta_Z)) \simeq \pi_*(\Omega_{m+1}(\Delta_W)/\Omega_{m+1}(\Delta_W - \Delta_Z)).$$

To compute the Chern classes of \mathcal{F} we adapt a classical argument that we learned in [Che13]. For every $i = 1, \dots, n$, we consider the bundle $\Omega_{m+1}(\Delta_W - a_i \Delta_{p+i, m+1}) = \Omega_{m+1} \otimes \mathcal{O}(\Delta_W - a_i \Delta_{p+i, m+1})$ on \mathcal{X}^{m+1} , where Ω_{m+1} is Hodge bundle on \mathcal{X}^{m+1} .

Consider the case $i = 1$ and in order to simplify the notation we will use Δ for the diagonal $\Delta_{p+1, m+1}$. We have exact sequences:

$$0 \rightarrow \Omega_{m+1}(\Delta_W - a_1 \Delta) \rightarrow \Omega_{m+1}(\Delta_W - (a_1 - 1) \Delta) \rightarrow \Omega_{m+1}(\Delta_W - a_1 \Delta)|_{\Delta} \rightarrow 0,$$

$$0 \rightarrow \Omega_{m+1}(\Delta_W - a_1 \Delta) \rightarrow \Omega_{m+1}(\Delta_W) \rightarrow \Omega_{m+1}(\Delta_W)|_{a_1 \Delta} \rightarrow 0,$$

$$0 \rightarrow \Omega_{m+1}(\Delta_W - (a_1 - 1) \Delta) \rightarrow \Omega_{m+1}(\Delta_W) \rightarrow \Omega_{m+1}(\Delta_W)|_{(a_1 - 1) \Delta} \rightarrow 0.$$

We define

$$F_1 := \pi_* \left(\frac{\Omega_{m+1}(\Delta_W - (a_1 - 1)\Delta)}{\Omega_{m+1}(\Delta_W - a_1\Delta)} \right), \quad F_0 := \pi_* \left(\frac{\Omega_{m+1}(\Delta_W)}{\Omega_{m+1}(\Delta_W - a_1\Delta)} \right),$$

$$\text{and } F_2 := \pi_* \left(\frac{\Omega_{m+1}(\Delta_W)}{\Omega_{m+1}(\Delta_W - (a_1 - 1)\Delta)} \right).$$

From the natural inclusions

$$\Omega_{m+1}(\Delta_W - a_1\Delta) \subset \Omega_{m+1}(\Delta_W - a_1\Delta) \subset \Omega_{m+1}(\Delta_W - (a_1 - 1)\Delta) \subset \Omega_{m+1}(\Delta_W)$$

and the fact that the sheaf $\frac{\Omega_{m+1}(\Delta_W - (a_1 - 1)\Delta)}{\Omega_{m+1}(\Delta_W - a_1\Delta)}$ is supported on Δ , we have the following exact sequence:

$$0 \rightarrow F_1 \rightarrow F_0 \rightarrow F_2 \rightarrow 0.$$

Since the divisor Δ has multiplicity a_1 at Δ_Z and $\Delta \notin \text{Supp}(\Delta_W)$, the fiber of F_1 at z_1 is

$$\left(K_X \left(\sum_{j=1}^p b_j w_j \right) \right) |_{z_1} \simeq (K_X |_{z_1})^{\otimes a_1} \otimes \mathcal{O}_{z_1} \left(\sum_{j=1}^p b_j w_j \right),$$

then we have that $F_1 \simeq \Omega_n^{\otimes a_1} \otimes \mathcal{O}_{\mathcal{X}^n}(\Delta_W)$.

Following this argument we can to construct a new filtration

$$0 \rightarrow F'_1 \rightarrow F_1 \rightarrow F'_2 \rightarrow 0,$$

where F'_1 is the sheaf defined as

$$F'_1 := \pi_* \left(\frac{\Omega_{m+1}(\Delta_W - (m_2 - 1)\Delta)}{\Omega_{m+1}(\Delta_W - (a_1 - 1)\Delta)} \right),$$

and we have that $F'_1 \simeq \Omega_n^{a_1 - 1} \otimes \mathcal{O}_{\mathcal{X}^n}(\Delta_W)$. In this way using a sequence of filtrations obtained by subtracting 1 successively to a_1 we get that

$$F_0 \simeq [\Omega_n^{\otimes a_1} \otimes \mathcal{O}_{\mathcal{X}^n}(\Delta_W)] \otimes [\Omega_n^{\otimes (a_1 - 1)} \otimes \mathcal{O}_{\mathcal{X}^n}(\Delta_W)] \otimes \cdots \otimes [\Omega_n \otimes \mathcal{O}_{\mathcal{X}^n}(\Delta_W)].$$

By subtracting 1 successively to all a_i we can reduce to signature $(1, 0, \dots, 0; -b_1, \dots, -b_p)$ to obtain the expression for the Chern class of \mathcal{F} as desired. \square

Applying Lemma 4.1 to the case of the stratum $\Omega\mathcal{M}_3(6; -2)$, we get that

$$\begin{aligned} c(\mathcal{F}) &= \prod_{j=1}^6 (1 + jK_2 + 2\Delta_{12}), \\ &= 1 + (21K_2 + 12\Delta_{12}) + (175K_2^2 + 210K_2\Delta_{12} + 60\Delta_{12}^2) \\ &\quad + (735K_2^3 + 1400K_2^2\Delta_{12} + 840K_2\Delta_{12}^2 + 160\Delta_{12}^3) + \dots \end{aligned}$$

By Proposition 2.4 we obtain that the class of $-E_2$ is

$$\begin{aligned} c(-E_2) &= 1 + (K_1 - \lambda_1) + (K_1^2 - \lambda_1 K_1 + \lambda_1^2 - \lambda_2) \\ &\quad + (K_1^3 - \lambda_1 K_1^2 + (\lambda_1^2 - \lambda_2) K_1 + 2\lambda_1 \lambda_2 - \lambda_3 - \lambda_1^3) + \dots \end{aligned}$$

So we obtain

$$\begin{aligned}
c_3(\mathcal{F} - E_2) &= (735K_2^3 + 1400K_2^2\Delta_{12} + 840K_2\Delta_{12}^2 + 160\Delta_{12}^3) \\
&\quad + (175K_2^2 + 210K_2\Delta_{12} + 60\Delta_{12}^2) \cdot (K_1 - \lambda_1) \\
&\quad + (21K_2 + 12\Delta_{12}) \cdot (K_1^2 - \lambda_1K_1 + \lambda_1^2 - \lambda_2) \\
&\quad + (K_1^3 - \lambda_1K_1^2 + (\lambda_1^2 - \lambda_2)K_1 + 2\lambda_1\lambda_2 - \lambda_3 - \lambda_1^3).
\end{aligned}$$

This is equal to the Chern class of $\mathcal{M}_3(6; -2)$ inside $A^3(\mathcal{M}_{3,2})$. We now simplify this expression. We first expand this sum, leading to the equality

$$\begin{aligned}
c_3(\mathcal{F} - E_2) &= 735K_2^3 + 175(K_1 - \lambda_1)K_2^2 + 21(K_1^2 - \lambda_1K_1 + \lambda_1^2 - \lambda_2)K_2 \\
&\quad + 1400\Delta_{12}K_2^2 + 210\Delta_{12}K_1K_2 - 210\Delta_{12}\lambda_1K_2 + 840\Delta_{12}^2K_2 \\
&\quad + K_1^3 - \lambda_1K_1^2 + 12\Delta_{12}K_1^2 + (\lambda_1^2 - \lambda_2)K_1 - 12\Delta_{12}\lambda_1K_1 + 60\Delta_{12}^2K_1 - \lambda_3 \\
&\quad + 2\lambda_1\lambda_2 - 12\Delta_{12}\lambda_2 - \lambda_1^3 + 12\Delta_{12}\lambda_1^2 - 60\Delta_{12}^2\lambda_1 + 160\Delta_{12}^3.
\end{aligned}$$

We now simplify this expression using the equalities recalled in the first part of Lemma 2.7. For example, the second equality of this lemma implies that the coefficient $60\Delta_{12}^2K_1$ is equal to $-60\Delta_{12}K_1^2$. Doing this for all the coefficient, this gives the expression

$$\begin{aligned}
c_3(\mathcal{F} - E_2) &= 735K_2^3 + 175(K_1 - \lambda_1)K_2^2 + 21(K_1^2 - \lambda_1K_1 + \lambda_1^2 - \lambda_2)K_2 \\
&\quad + (882K_1^2 - 162\lambda_1K_1 + 12(\lambda_1^2 - \lambda_2))\Delta_{12} \\
&\quad + K_1^3 - \lambda_1K_1^2 + (\lambda_1^2 - \lambda_2)K_1 - \lambda_3 + 2\lambda_1\lambda_2 - \lambda_1^3.
\end{aligned}$$

We now forget the two marked points to obtain the λ -class of $\mathcal{M}_3(6; -2)$ in the Picard group of \mathcal{M}_3 . By forgetting the second point and using the formulas of the second part of Lemma 2.7 we obtain

$$\begin{aligned}
\pi_{2,*}(c_3(\mathcal{F} - E(2))) &= 735\kappa_2 + 175(K_1 - \lambda_1)\kappa_1 + 21(K_1^2 - \lambda_1K_1 + \lambda_1^2 - \lambda_2)\kappa_0 \\
&\quad + 882K_1^2 - 162\lambda_1K_1 + 12\lambda_1^2 - 12\lambda_2.
\end{aligned}$$

And finally by forgetting the first point we get

$$\pi_{1,*} \circ \pi_{2,*}(c_3(\mathcal{F} - E(2))) = 882\kappa_1 - 162\kappa_0\lambda_1 + 175\kappa_0\kappa_1 - 21\kappa_0^2\lambda_1 + 21\kappa_0\kappa_1.$$

Hence using that $\kappa_0 = 4$ and $\kappa_1 = 12\lambda_1$ we obtain that the class of $\pi_{1,*} \circ \pi_{2,*}(c_3(\mathcal{F} - E(2)))$ in $\text{Pic}(\mathcal{M}_3)$ is

$$[\pi_{1,*} \circ \pi_{2,*}(c_3(\mathcal{F} - E(2)))] = 19008\lambda_1.$$

The degeneracy locus of the map $\phi: E_2 \rightarrow \mathcal{F}$ contains the locus that we are interested in of curves X together with points w and z such that there exists a differential with divisor $6z - 2w$ and a locus supported in the diagonal Δ . So in order to obtain the class of $\mathcal{M}_3(6; -2)$ in the Picard group of \mathcal{M}_3 it remains to subtract this contribution. This is in general a delicate task and we refer to [Har84, Cuk89, Est16] for some examples.

At a point (X, z, z) of the diagonal, the evaluation map restricts to

$$\phi_z: H^0(X, K_X(2z)) \rightarrow H^0(X, K_X(2z)/K_X(-4z)).$$

This has generically rank 4 as expected and has rank 3 exactly if there is an abelian differential on X which has a zero of order 4 at z . The class of this locus is classical but we recall it for completeness. We have

$$c(\mathcal{F}) = 1 + 10K_1 + 35K_1^2 + \dots .$$

Hence

$$c_2(\mathcal{F} - E) = \lambda_1^2 - 10\lambda_1\kappa_1 + 35K_1^2 .$$

When pushing down to \mathcal{M}_3 we obtain

$$\begin{aligned} \pi_{1,*}(c_2(\mathcal{F} - E)) &= 35\kappa_1 - 10\lambda_1\kappa_0 \\ &= 35 \cdot 12\lambda_1 - 10 \cdot 4\lambda_1 \\ &= 380\lambda_1 , \end{aligned}$$

where we used again that $\kappa_1 = 12\lambda_1$ and $\kappa_0 = 4$.

Now we compute the multiplicity of this locus. Consider a point (X, z) where z is such that there exists a differential ω on X whose divisor is $4z$. Locally the vector bundle E_2 is generated by four sections. Let u is a local coordinate of X at z , we can take these sections to be $\eta_t = \frac{du}{u^2}$, the family of differentials $u^2\eta_t$, a family of differentials vanishing at order 1 or 2 at z depending if X is non hyperelliptic or hyperelliptic respectively and finally a family of the form $\omega_t := u^2(u^4 + s)\omega_0$ where s is a function of the base vanishing at X . So the locus of pointed curves (X, z) is given by the scheme given by the two by two minors of the matrix

$$\begin{pmatrix} 1 & u^2 & u^3h & u^2(u^4 + s) \\ 0 & 2u & \partial_u(u^3h) & 6u^5 + 2us \end{pmatrix} ,$$

where h depends if the point is Weierstraß or not. It is easy to check that the column where it appears the function h will not contribute to the multiplicity. In fact, the multiplicity of this point is given by the order of vanishing of s . In order to compute this order note that ω_t leads to a family of multi-scale differentials after blowing up the diagonal Δ . Note that the equation of the node is then given by $xy = f$ where f is the function defining the diagonal. Moreover, the function s is a rescaling parameter for this family of multi-scale differentials as defined in [BCG⁺19, Section 7.1]. More precisely, according to Equation (7.1) and Definition 7.2 of [BCG⁺19], we have $s = f^5$, where the 5 is the prong number. This implies that the multiplicity of the considered locus is 5.

Summing up all the computations of this section, we obtain that the class of $\mathcal{M}_3(6; -2)$ in the Picard group of \mathcal{M}_3 is

$$(11) \quad [\mathcal{M}_3(6; -2)] = 19008\lambda_1 - 5 \cdot 380\lambda_1 = 17108\lambda_1 .$$

5. CLASS OF THE DIVISOR $\overline{\mathcal{M}}_3(6; -2)$

In this section we compute the class of the locus $\overline{\mathcal{M}}_3(6; -2)$ in the rational Picard group $\text{Pic}(\overline{\mathcal{M}}_3) \otimes \mathbb{Q}$, proving Theorem 1.1. We first give in Section 5.1 some general facts on degenerating families of curves and differentials. Then we solve in Section 5.2 some enumerative problems related to differentials of the second kind. And finally in Section 5.3 we use this information to compute the class $[\mathcal{M}_3(6; -2)]$ using the method of test curves.

5.1. General facts on degenerating families. First we consider the problem of finding out how many fibres are isomorphic in some families of semi-stable curves. These families naturally appear in the context of plumbing of differentials. To introduce them, we recall from [BCG⁺18] that a *plumbing fixture* is the family $\pi_a: \mathbb{V}_a \rightarrow \Delta_1$ of cylinders degenerating to a node

$$\mathbb{V}_a(t) = \{(u, v, t) \in \Delta_1^3 : uv = t^a\}$$

where $a \in \mathbb{Z}_{>0}$ and the projection is given by $\pi_a(u, v, t) = t$. We will look at the families of curves obtained by gluing the plumbing fixture at the nodal points of a trivial family of pointed curves. In particular, the annuli on the curves at which we glue the plumbing fixtures are constant. These families are special cases of the families appearing in the plumbing setup of [BCG⁺18, p. 2394].

Lemma 5.1. *Let X_0 be a curve with one node n and X the curve obtained by gluing a projective line E at n . We denote by n_1 and n_2 the nodes of X . Let (B, p) be a smooth curve with parameter t at p and let $f: X \rightarrow (B, p)$ be the family obtained by gluing the plumbing fixtures $x_1 y_1 = t^{a_1}$ and $x_2 y_2 = ct^{a_2}$ with $|c| = 1$ at the nodes n_i of X . Then the induced map $\mu_f: B \rightarrow \mathcal{M}_g$ is a cover ramified at p of degree $a_1 + a_2$ on its image.*

Proof. The proof has two parts. First we are going to give a condition so that two curves of the basic family are isomorphic near X . Then we are going to count the number of curves that satisfy that criterion in the hypotheses of the lemma.

Let X be a semi-stable curve satisfying the hypotheses of Lemma 5.1 with exceptional component E . Consider the family $\mathcal{Y} \rightarrow \Delta^2 = \{(t_1, t_2) : |t_i| < 1\}$ such that the curve above the point (t_1, t_2) is obtained by gluing the plumbing fixtures $\mathbb{V}_a(t_i)$ at n_i . Moreover, we assume that the plumbing is such that x_i are local equations of E in \mathcal{Y} and $y_1 y_2 = 1$. Consider the two curves X_{θ_1, θ_2} and $X_{\vartheta_1, \vartheta_2}$ above the parameters $(r_1 \exp(i\theta_1), r_2 \exp(i\theta_2))$ and $(r_1 \exp(i\vartheta_1), r_2 \exp(i\vartheta_2))$. We now show that these curves are isomorphic if and only if $\theta_1 + \theta_2 = \vartheta_1 + \vartheta_2$.

Let us study when the identity of a neighborhood of a complement of $x_i = 0$ in the curves X_{θ_1, θ_2} and $X_{\vartheta_1, \vartheta_2}$ can be extended to an isomorphism $\varphi: X_{\theta_1, \theta_2} \rightarrow X_{\vartheta_1, \vartheta_2}$. The identity can be extended through the annuli $x_i y_i = t_i$ by the functions

$$\phi_i: (x_i, y_i) \mapsto (x_i, \exp(i(\theta_i - \vartheta_i))y_i).$$

The functions ϕ_i can be extended to an holomorphic function on E if and only if ϕ_1 coincides with ϕ_2 on E , that is

$$\exp(i(\theta_1 - \vartheta_1))y_1 = \exp(-i(\theta_2 - \vartheta_2))y_2 = \exp(-i(\theta_2 - \vartheta_2))y_1^{-1}.$$

From that equation it follows directly that the curves X_{θ_1, θ_2} and $X_{\vartheta_1, \vartheta_2}$ are isomorphic if and only if $\theta_1 - \vartheta_1 + \theta_2 - \vartheta_2 = 0$.

Now we fix two integers $a_1, a_2 \geq 1$ as in lemma 5.1 and a real number $r > 0$. We introduce the curve

$$C_{a_1, a_2, c} = \{(a_1 \theta, c + a_2 \theta) : \theta \in \mathbb{S}^1\} \subset \mathbb{S}^1 \times \mathbb{S}^1.$$

It follows from the first part of the proof that it suffices to show that the antidiagonal $\mathcal{A} = \{(\theta, -\theta) : \theta \in \mathbb{S}^1\}$ in $\mathbb{S}^1 \times \mathbb{S}^1$ and the locus $C_{a_1, a_2, c}$ have $a_1 + a_2$ points of intersection. Since for every c the curve $C_{a_1, a_2, c}$ is a translation of $C_{a_1, a_2, 0}$ the number of intersections

does not depend on c . Assume first that a_1 and a_2 are prime to each other. The intersections between Δ and $C_{a_1, a_2, 0}$ are given by the points $(k/(a_1 + a_2), -k/(a_1 + a_2))$ with $k \in \{0, \dots, a_1 + a_2 - 1\}$. Hence there are $a_1 + a_2$ points of intersection between the loci Δ and $C_{a_1, a_2, 0}$. If $d = \gcd(a_1, a_2) > 1$ then the curves Δ and $C_{a_1, a_2, 0}$ have $a_1/d + a_2/d$ points of intersection, each of multiplicity d . \square

The following result will be very useful in order to compute some intersection numbers in Section 5.3.

Lemma 5.2. *Let $(X; z, w; \omega)$ be a twisted differential of type $(a; -b)$ of genus $g \geq 2$. Then the underlying pointed curve $(X; z, w)$ is not one of the curves pictured in Figure 2 where E is a rational curve.*

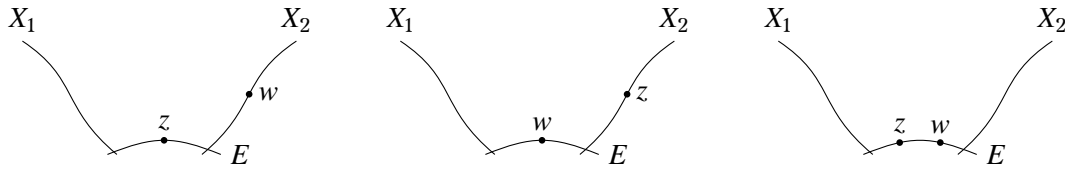


FIGURE 2. The stable pointed curve $(X; z, w)$ with one projective bridge.

Before completing the proof, we remark that it seems to be a general principle that special points related to differentials do not easily converge to a separating node. On the other hand, we will see in Section 5.2 that special points can easily converge to separating nodes. For an other instance of this principles, the reader can look at [Gen21].

Proof. In these three cases the restriction of ω to E has at least two poles. Hence Theorem 1.2 of [GT21] implies that the residues of these nodes are different from zero. Hence the global residue condition, that is recalled in Definition 2.1, can not be satisfied on any of these curves. \square

Finally, we give the form of the curve underlying a multi-scale differential of second kind with a unique zero in genus 1.

Lemma 5.3. *The multi-scale differentials $(X; z, w_1, \dots, w_p, \omega; \sigma; \leq)$ in the boundary of the locus $\Omega\mathcal{R}_1(a; -b_1, \dots, -b_p)$ of differentials of the second kind are such that X is either irreducible (with one node) or is two rational curves which are glued together at two nodes.*

Proof. If X was a curve not of one type given in Lemma 5.3, then either X has a rational component with marked points glued at the rest of the curve at one node or it is a chain of rational components of length strictly greater than 2.

In the first case the differential on the rational curve has exactly one zero. Hence by [GT21] it has some poles with non zero residues. This implies that this differential can not be in the limit of a family of differential of $\Omega\mathcal{R}_1(a; -b_1, \dots, -b_p)$ as proved in [Mul19, Theorem 1.1].

In the second case, consider the component X_1 of X which contains the zero z of the multi-scale differential. Then the component X_1 is the unique local minimum for the full order \leq as shown in [BCG⁺18, Lemma 3.9]. This implies that the differential $\omega|_{X_1}$

has two poles at the nodal points of X_1 . On the adjacent components X_2 and X_3 the differentials $\omega|_{X_i}$ have a zero at the corresponding nodal point. Hence at the other nodal points the differentials on $\omega|_{X_2}$ and $\omega|_{X_3}$ have another zero. Indeed, suppose that $\omega|_{X_2}$ has a pole at the other nodal point of X_2 . Then by [GT21, Theorem 1.2] the form $\omega|_{X_2}$ has at least two poles with nonzero residues, one of which being a marked pole, giving a contradiction to [Mul19, Theorem 1.1]. This implies that there are both higher for \leq than some other components of X . Hence there is another local minimum, contradicting the uniqueness of the local minimum for \leq . \square

5.2. Some enumerative problems. Let us begin with a proposition giving the number of differentials with a unique zero and a unique pole on a curve of genus 2. This is essentially known (see [CT16, Proposition 2.2] or [Mul17, Section 2.6]) but we recall since it is only implicitly stated in the cited articles.

Proposition 5.4. *For every $a \geq 2$, the map $\pi: \mathbb{P}\Omega\mathcal{M}_2(a+2; -a) \rightarrow \mathcal{M}_2$ is an unramified cover of degree $2(a+2)^2a^2 - 18$.*

Proof. Let C be a general curve of genus 2 and consider the map

$$f: C^2 \rightarrow \text{Pic}^2(C) : (p_1, p_2) \mapsto (a+2)p_1 - ap_2.$$

By [CT16] and [Mul17] we know that f is a finite map of degree $2(a+2)^2a^2$. Moreover, this map is ramified at the loci

$$\Delta = \left((p_1, p_2) \in C^2 : p_1 = p_2 \right)$$

and

$$\mathcal{K} = \left((p_1, p_2) \in C^2 : p_1 = \iota(p_2) \right),$$

where ι is the hyperelliptic involution. Note that $\Delta \cap \mathcal{K}$ is the set of the six Weierstraß points of C . Let us consider the preimage by f of the canonical bundle of C . If a pair (p_1, p_2) is in the preimage of K_C belongs to Δ , then $(a+2-a)p_1$ is canonical, so p_1 is a Weierstraß point. Second, if a pair (p_1, p_2) of the preimage of K_C belongs to \mathcal{K} , then there are differentials ω_1 and ω_2 whose respective divisors are $(a+2)p_1 - ap_2$ and $p_1 + p_2$. Then $\eta = \omega_1\omega_2^a$ is a $(a+1)$ -differential with a unique zero of order $2a+2$. The $(a+1)$ -differential η exists on a general curve if and only if η is the $(a+1)$ th power of an abelian differential with a double zero. This implies that $p_1 = p_2$ is a Weierstraß point.

We now compute the degree of ramification of f at the Weierstraß points as in [Mul17]. Let p be a Weierstraß point and (ω_1, ω_2) generators of $H^0(C, K_C)$ such that $\text{ord}_p(\omega_1) = 0$ and $\text{ord}_p(\omega_2) = 2$. The derivative of f at p is of the form

$$Df(p) = \begin{pmatrix} 1 & 1 \\ t_1^2 & t_2^2 \end{pmatrix}$$

where t_i are functions which vanish at p at order 1. Hence the determinant of this matrix vanishes at order 2 at p . It follows that the order of branching at p is equal to 2. Hence there are 18 pairs in the preimage of the canonical divisor K_C that do not correspond to solutions in the stratum $\Omega\mathcal{M}_2(a+2; -a)$ but to abelian differentials with a single zero order 2.

It remains to show that the cover is not branched. Suppose it is branched, then the branching point corresponds to a twisted differential where the underlying curve is a

genus 2 curve with a rational curve E glued at a point. The curve E contains the zero of order $a + 2$ and the pole of order $-a$. Hence the restriction of the differential to E has two poles and a unique zero, hence the residues at the poles are non-zero. Hence this twisted differential does not satisfy the global residual condition of Definition 2.1 and is not smoothable. \square

We will now deduce an important consequence of this result for the locus of differentials of second kind on elliptic curves.

Corollary 5.5. *The degree of the map $\pi: \mathbb{P}\Omega\mathcal{R}_1(a + 2; -a, -2) \rightarrow \mathcal{M}_{1,1}$ forgetting the poles is equal to $(a + 2)^2 + a^2 - 10$ for $a \geq 3$ and 5 for $a = 2$.*

Proof. Let us consider a stable curve X of genus 2 which is the union of two elliptic curves attached at one node. We describe all the multi-scale differentials on a curve X semi-stably equivalent to X . By Lemma 5.2 the pointed curves which can appear in such multi-scale differentials are of the form pictured in Figure 3.

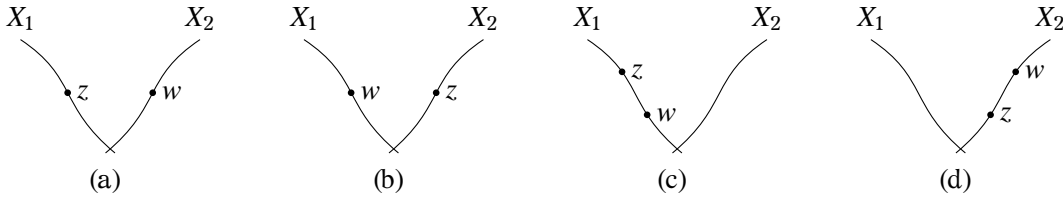


FIGURE 3. The pointed curves $(X; z, w)$ which can appear in multi-scale differentials at the boundary of the stratum $\Omega\mathcal{M}_2(a + 2; -a)$.

In case (a) the differential ω_1 on X_1 is in the stratum $\Omega\mathcal{M}_1(a + 2; -a - 2)$ and the differential ω_2 on X_2 is in the stratum $\Omega\mathcal{M}_1(a; -a)$. Hence there are $(a + 2)^2 - 1$ different differentials on X_1 and $a^2 - 1$ different differentials on X_2 . Since there is clearly only one class of prong-matching there are $((a + 2)^2 - 1)(a^2 - 1)$ different multi-scale differentials of this type. Moreover, the space of multi-scale differentials is smooth at this point, hence $((a + 2)^2 - 1)(a^2 - 1)$ degenerate to this type. The type (b) is similar. Hence the intersection number with the ones of type (c) (or (d)) is equal to

$$\begin{aligned} D &= ((a + 2)^2 a^2 - 9) - (((a + 2)^2 - 1)(a^2 - 1)), \\ &= 2a^2 + 4a - 6, \\ &= (a + 2)^2 + a^2 - 10. \end{aligned}$$

Since the space of multi-scale differentials is smooth at these points and there is a unique differential on X_2 (in the case (c)), this implies that the number of distinct differentials of the second kind in $\Omega\mathcal{M}_1(a + 2; -a, -2)$ is $(a + 2)^2 + a^2 - 10$ for $a \geq 3$. The case $a = 2$ is similar taking into account that there is a symmetry since both poles have the same order. \square

We now compute the degree of the map $\pi: \mathbb{P}\Omega\mathcal{R}_1(6; -2, -2, -2) \rightarrow \mathcal{M}_{1,1}$ proving Theorem 1.3. In order to illustrate the ideas of the proof in easier context, we first compute

the degrees of the maps $\pi: \mathbb{P}\Omega\mathcal{R}_1(4; -2, -2) \rightarrow \mathcal{M}_{1,1}$ and $\pi: \mathbb{P}\Omega\mathcal{R}_1(5; -3, -2) \rightarrow \mathcal{M}_{1,1}$ (that is already known by Corollary 5.5).

Lemma 5.6. *The degree of the map $\pi: \mathbb{P}\Omega\mathcal{R}_1(4; -2, -2) \rightarrow \mathcal{M}_{1,1}$ forgetting the poles is 5 and the degree of the map $\pi: \mathbb{P}\Omega\mathcal{R}_1(5; -3, -2) \rightarrow \mathcal{M}_{1,1}$ is 24.*

The proof of this result is by degeneration in the moduli space of multi-scale differentials. The results that we use here are recalled in Section 2.1.

Proof. We first consider the case of the locus $\Omega\mathcal{R}_1(4; -2, -2)$ of differentials of the second kind of type $(4; -2, -2)$. By Lemma 5.3 there are 3 types of pointed curves on which there can exist a multi-scale differential in the closure of this locus. This is summarised in Table 1 that we first explain in details. Note that Tables 2 and 3 have the same entries.

| | | | |
|-------------------------|---|---|---|
| | | | |
| Pointed curve | | | |
| # twisted differentials | 1 | 1 | 1 |
| # prong-matching | 1 | 1 | 2 |
| Local degree | 1 | 2 | 2 |
| Symmetries | 1 | 1 | 2 |
| Total count | 1 | 2 | 2 |

TABLE 1. Summary of the case $\Omega\mathcal{R}_1(4; -2, -2)$.

The first row gives the pointed stable curves on which there may exist a multi-scale differential. This is only given up to permuting the poles of the same order, for example in the second column, the case where w_1 and w_2 are permuted gives the same differential. The numbers κ_i are the prong numbers at the corresponding node.

The second row gives the number of twisted differentials that exists on the pointed curve under consideration.

The third row gives the number of classes of prong-matching. In the cases we will consider here, the multi-scale differentials have only two nodes and two levels. Hence this number is simply given by $\gcd(\kappa_1, \kappa_2)$.

The fourth row gives the local degree of the stabilisation map $\mathcal{M}_{1,m} \rightarrow \mathcal{M}_{1,1}$ restricted to the smooth pointed curves underlying the differentials obtained by smoothing the multi-scale differential under consideration. We show that this degree is given by Lemma 5.1. Note that if there is more than one class of prong-matching, the degree is given for each choice of prong-matching but it does not depend on this choice.

According to the fourth row, on a smooth curve near the boundary of $\mathcal{M}_{1,1}$ the number of pointed differentials on the curves which degenerate to the multi-scale differential under consideration is equal to the degree. However, it happens that these pointed differentials only differ by a permutation of the marked points. The fifth row gives the order of the group generated by these permutations of the marked points.

Finally, in the last row we compile this information to give the number of differentials of type μ on a smooth curve which degenerate to a multi-scale differential such that the underlying pointed curve is the one of the first row.

We now return to the case of $\Omega\mathcal{R}_1(4; -2, -2)$. Consider the irreducible pointed curve pictured in the first column. Any multi-scale differential on this pointed curve has two single poles at the node. Hence the pull-back of the differential on the normalisation of the curve lies in the stratum $\Omega\mathcal{M}_0(4; -2, -2, -1, -1)$ and the residues at the poles of order 2 are equal to 0. There is a unique such differential whose flat representation is shown in Figure 4. The reader can consult [MT02, Section 1] and [Boi15] for the correspondence between flat surfaces and differentials respectively in the holomorphic and meromorphic case. Moreover, the number of prong matching and the local degree

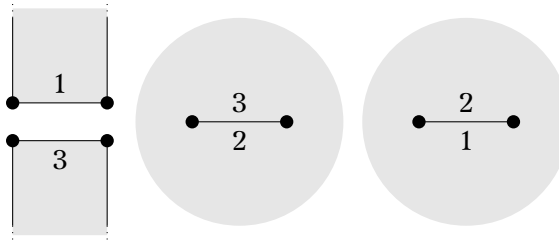


FIGURE 4. The differential of $\Omega\mathcal{M}_1(4; -2, -2, -1, -1)$ such that the residues are $(0, 0, 1, -1)$.

are clearly equal to 1. Hence on a smooth curve near the boundary of $\mathcal{M}_{1,1}$ there is a unique differential which degenerate to a multi-scale differential with this underlying pointed curve.

We now consider the curve pictured in the second column of Table 1. First note that the prong numbers have to be equal to 1, since otherwise there would exist no full order \leq . Hence the differential on the upper component is in the stratum $\Omega\mathcal{M}_0(0, 0; -2)$ and the differential on the lower component is in the stratum $\Omega\mathcal{M}_0(4; -2, -2, -2)$ and the residue vanishes at the marked pole. There is clearly a unique differential in $\Omega\mathcal{M}_0(0, 0; -2)$. Moreover [CC19] show that there is a unique differential satisfying the second conditions which is shown in Figure 5. Since there is a unique class of prong-matching, there is a

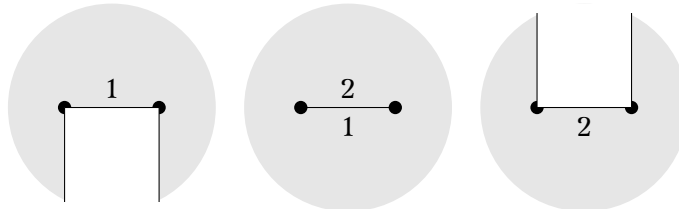


FIGURE 5. The differential of $\Omega\mathcal{M}_1(4; -2, -2, -2)$ with residues equal to $(0, 1, -1)$.

unique multi-scale differential ω with this underlying pointed curve. We can plumb this

multi-scale differential with a plumbing fixture to constant families of pointed curves. Indeed the addition of a modification differential does not change the order of the singularities of the differential of the top component. Hence by Lemma 5.1 in the case $a_1 = a_2 = 1$ there exists a neighborhood of the boundary of $\overline{\mathcal{M}}_{1,1}$ on which there are two differentials near ω . Moreover, as recalled in Section 2.1, in a neighborhood of this multi-scale differential the restriction of the differential on the top component looks like ω_0 and on the bottom component like $t\omega_{-1}$. The curves are isomorphic for the parameters t and $-t$ and hence the differentials on these curves are distinct.

We now consider the case pictured in the third column. Since the residues of the poles of order -2 vanish, Theorem 1.2 of [GT21] implies that $\kappa_1 = \kappa_2 = 2$. By proposition 2.3 of [CC19] there is a unique element in $\Omega\mathcal{M}_0(1, 1; -2, -2)$ such that the residues at the poles vanish. Moreover there are 2 classes of prong-matching which give two disjoint families of multi-scale differentials. Locally at the nodes n_i of the pointed curve, the first family \mathcal{X}_1 is given by $x_i y_i = t$ and the second one \mathcal{X}_2 is given by $x_i y_i = (-1)^i t$ for $i = 1, 2$. Moreover we know that in a neighborhood of the multi-scale differential the form on the top component looks like ω_0 and on the bottom component like $t^2\omega_{-1}$. Hence the differentials for t and for $-t$ are equals to each other. Hence on the isomorphic curves above t and $-t$ both differentials are isomorphic. Hence there is a unique differential degenerating to each multi-scale differential on this pointed curve.

We conclude that the number of distinct differentials in $\Omega\mathcal{R}_1(4; -2, -2)$ on a smooth curve of genus 1 is $1 + 2 + 2 = 5$.

We now deal with the case of the locus $\Omega\mathcal{R}_1(5; -3, -2)$. There are four different types of pointed curves on which there can exist a multi-scale differential in the limit of the locus $\Omega\mathcal{R}_1(5; -3, -2)$. These pointed curves are pictured in the first row of Table 2.

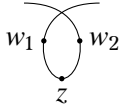
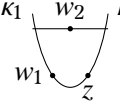
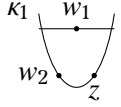
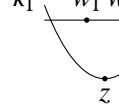
| | | | | |
|-------------------------|---|---|--|---|
| Pointed curve |  |  |  |  |
| # twisted differentials | 4 | 2 | 1 | 2 |
| # prong-matching | 1 | 1 | 1 | 1 |
| Local degree | 1 | 2 | 3 | 5 |
| Symmetries | 1 | 1 | 1 | 1 |
| Total count | 4 | 4 | 6 | 10 |

TABLE 2. Summary of the case $\Omega\mathcal{R}_1(5; -3, -2)$.

Consider the irreducible curve of the first column. In this case the differential has simple poles at the nodal points. So its pull-back lies in the stratum $\Omega\mathcal{M}_1(5; -3, -2, -1, -1)$ and its residues at the non simple poles vanish. By Proposition 3.8 of [CC19] there are precisely 2 non isomorphic such differentials represented in Figure 6. Moreover the simple poles can be distinguished by the fact that one is next to the pole of order -2 and the other next to the pole of order -3 . Hence we obtain 4 such multi-scale differentials. It is easy to see that each of these differentials can be smoothed in a unique way.



FIGURE 6. The two differentials of $\Omega\mathcal{M}_1(5; -3, -2, -1, -1)$ with residues equal to $(0, 0, 1, -1)$.

Now consider the curve in the second column of Table 2, when the pole of order -2 is on the top component. The differential on the top component is the unique differential in $\Omega\mathcal{M}_0(0, 0; -2)$. The differential on the bottom component lies in the meromorphic stratum $\Omega\mathcal{M}_0(5; -3, -2, -2)$ with a residue equal to 0 at the pole of order -3 . Again [CC19, Proposition 3.8] implies that there are 2 non isomorphic such differentials. As before the local degree of each family is 2 and there are two distinct differential on a smooth curve which degenerate to the considered pointed curve.

Now consider the curve in the third column of Table 2, when the pole of order -3 is on the top component. The differential on the top component is in the stratum $\Omega\mathcal{M}_0(1, 0; -3)$. The differential on the bottom component is in the stratum $\Omega\mathcal{M}_0(5; -3, -2, -2)$ with a residue equal to 0 at the marked pole of order -2 . There is a unique such differential. Note that the prong numbers are equal to 1 and 2. Hence there are two such twisted differential depending on which node we put the distinct prong numbers. There is only one class of prong-matching. Since locally at the nodes the families are given by the equations $x_1y_1 = t^2$ and $x_1y_1 = t$, and that we can plumb the multi-scale differential without modifying the top curve away from the nodes, we deduce from Lemma 5.1 that each family is a cover of order 3 to $\mathcal{M}_{1,1}$. Hence there are 6 differentials on a smooth curve which arise from this case.

Finally, we consider the fourth pointed curve where both poles are on the top component. The differential on this component is the locus $\Omega\mathcal{R}_0(2, 1; -2, -3)$. Again [CC19] gives that there is a unique such differential. The differential on the bottom component is the unique differential in $\Omega\mathcal{M}_0(5; -2, -3)$. Since there are two ways to assign the prong numbers at the nodes, there are two twisted differentials on this pointed curve. Note that there is a unique class of prong-matching, hence there are two distinct multi-scale differentials on this pointed curve. Again each family is locally a cover of order 5 to $\mathcal{M}_{1,1}$. Hence there are 10 differentials which come from this case.

Summing up all the contributions we conclude that there are 24 non isomorphic differentials in $\Omega\mathcal{R}_1(5; -2, -3)$ on a smooth curve of genus 1. \square

We now prove Theorem 1.3 by arguments similar to the ones used to prove Lemma 5.6. Recall that this theorem says that the degree of $\pi: \mathbb{P}\Omega\mathcal{R}_1(6; -2, -2, -2) \rightarrow \mathcal{M}_{1,1}$ is 7. Note that by Theorem 1.2, the dimension of $\mathbb{P}\Omega\mathcal{R}_1(6; -2, -2, -2)$ is equal to the dimension of $\mathcal{M}_{1,1}$, hence it make sense to compute the degree of the forgetful map.

Proof of Theorem 1.3. According to Lemma 5.3 there are 4 types of pointed curves on which there can exists a multi-scale differential in the closure of $\Omega\mathcal{R}_1(6; -2, -2, -2)$.

These curves are shown in the first row of Table 3 (the entries of this table are explained after Table 1).

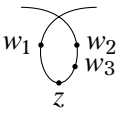
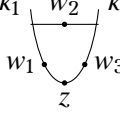
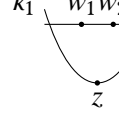
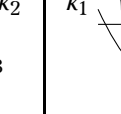
| |  |  |  |  |
|-------------------------|---|---|--|---|
| Pointed curve | | | | |
| # twisted differentials | 1 | 1 | 1 | 1 |
| # prong-matching | 1 | 1 | 2 | 3 |
| Local degree | 1 | 2 | 2 | 2 |
| Symmetries | 1 | 1 | 2 | 3 |
| Total count | 1 | 2 | 2 | 2 |

TABLE 3. Summary of the case $\Omega\mathcal{R}_1(6; -2, -2, -2)$.

Let us consider the pointed curves of the first column. The pull-back of the differential on the normalisation lies in the stratum $\Omega\mathcal{M}_0(6; -2, -2, -2, -1, -1)$ with all the residues of the poles of order -2 equal to zero. There is a unique such differential, and it is not difficult to show that it leads to a unique multi-scale differential on such pointed curve. Hence there is a unique differential on a smooth curve degenerating to this multi-scale differential.

Consider the pointed curve in the second column. The differential on the top component is the unique one in the stratum $\Omega\mathcal{M}_0(0, 0; -2)$. The differential on the bottom component is in $\Omega\mathcal{M}_0(6; -2, -2, -2, -2)$ with two poles having zero residue. There is a unique such twisted differential by [CC19]. Moreover, since there is a unique class of prong-matching, there is a unique multi-scale differential on this pointed curve. The local equation of the family at the nodes is $x_i y_i = t$, and we can plumb the family so that the underlying curve satisfies the hypotheses of Lemma 5.1, so by Lemma 5.1 the local degree of the family of pointed curves is 2. Since the family of differentials near the top component is given by ω_0 and the bottom by $t\omega_{-1}$, the differentials on the isomorphic curves for parameters t and $-t$ are distinct. Hence there are two differentials on a smooth genus 1 curves converging to this multi-scale differential.

Consider the case of pointed curves given in the third column of Table 3. The differential on the top component is the unique differential in $\Omega\mathcal{R}_0(1, 1; -2, -2)$. The differential on the bottom component is in the stratum $\Omega\mathcal{M}_0(6; -2, -3, -3)$ with two poles having zero residue. Proposition 2.3 of [CC19] implies that there is a unique such twisted differential. Since the prong-numbers are both equal to 2, there are 2 classes of prong-matchings. For each class the degree of the restriction of the stabilisation map to this family is 2. But since near the bottom differential the family looks like $t^2\omega_{-1}$, the 2 differentials coincide (the marked poles are permuted). Hence there are 2 differentials on a smooth curve which degenerate to the multi-scale differentials in this case.

In the last pointed curve of Table 3, the restriction ω_0 of the differential to the top component is the unique element in the locus $\Omega\mathcal{R}_0(2, 2; -2, -2, -2)$. Note that the top differential in $\Omega\mathcal{R}_0(2, 2; -2, -2, -2)$ has an automorphism group equal to $\mathbb{Z}/3\mathbb{Z}$. This can be done by checking directly that $\omega_0 = \frac{z^2}{(z^3-1)^2} dz$ or showing at the flat picture of it

given in Figure 7. The differential on the bottom component is the unique element in $\Omega\mathcal{M}_0(6; -4, -4)$. There is a unique such twisted differential but there are 3 classes of prong-matchings. Here we have a new phenomenon occurring. Indeed the automorphism group of ω_0 acts on the classes of prongs and hence on the multi-scale differentials. One class of prongs is represented in Figure 7 and the acts of the automorphism group consist of permuting cyclically the polar domains. Hence there is only one class prong-matching modulo isomorphism. For the smoothing of this multi-scale differential the degree of the stabilisation map is 2. It is easy to see that the 2 differentials on isomorphic curves are indeed distinct. Hence there are 2 differentials on a smooth curve degenerating to this multi-scale differential.

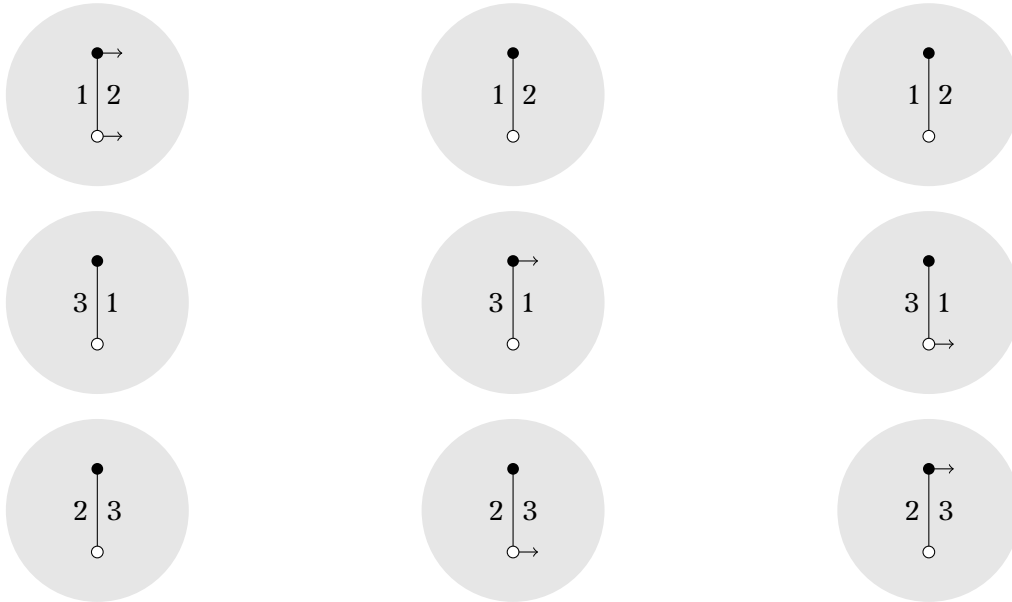


FIGURE 7. The three element in the same class of prong-matchings in the case $\Omega\mathcal{R}_1(6; -2, -2, -2)$.

Summing up all the cases we obtain that there are 7 non isomorphic differentials in the locus $\mathbb{P}\Omega\mathcal{R}_1(6; -2, -2, -2)$ on a general curve of genus 1.

To conclude the proof, it suffices to show that the map $\pi: \mathbb{P}\Omega\mathcal{R}_1(6; -2, -2, -2) \rightarrow \mathcal{M}_{1,1}$ is unramified. To prove this, it suffices to show that there exit no twisted differentials of type (6; -2, -2, -2) of the second kind on a curve semi-stably equivalent to a smooth curve of genus 1. This follows from [Mul19, Theorem 1.1] by the fact that every twisted differential on such curve has a non-zero residue either at a node or at a smooth pole. \square

To conclude this section on enumerative problems related to differentials, we compute the degree of the map $\pi: \mathbb{P}\Omega\mathcal{R}_2(6; -2, -2) \rightarrow \mathcal{M}_2$, which is finite by Theorem 1.2.

Proposition 5.7. *The degree of the map $\pi: \mathbb{P}\Omega\mathcal{R}_2(6; -2, -2) \rightarrow \mathcal{M}_2$ is 644.*

The proof of this proposition is by degeneration in the moduli space of multi-scale differentials.

Proof. Let us consider the stable curve X which is union of two general elliptic curves X_1 and X_2 attached to a point q . We denote by z the zero of order a , by w_1 and by w_2 the poles of order 2. We compute the number of multi-scale differentials on X .

There are six types of pointed curves $(X; z, w_1, w_2)$ which can have a multi-scale differential of type $(6; -2, -2)$ of the second kind. The ones with $z \in X_1$ are shown in Figure 8 and the other stable curves are symmetric to these ones with $z \in X_2$.

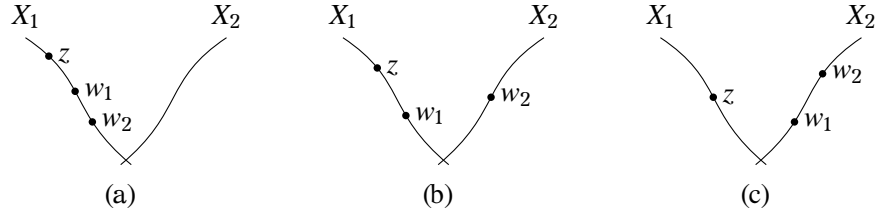


FIGURE 8. The pointed curves (X, z, w_1, w_2) underlying multi-scale differentials in the closure of $\Omega\mathcal{R}_2(6; -2, -2)$ (up to permutation of the points).

In case (a), the differential ω_1 on X_1 is in the locus $\Omega\mathcal{R}_1(6; -2, -2, -2)$. Theorem 1.3 gives that there are 7 non isomorphic such differentials. Moreover, the differential ω_2 on X_2 is the unique holomorphic differential on this elliptic curve. Since there are three ways of pasting these differentials together (one for each pole of the differentials ω_1), then there are $d_a = 21$ such twisted differentials. Moreover since there is a unique class of prong-matchings there are 21 multi-scale differentials of this type.

In case (b), the differential over X_1 is in the locus $\Omega\mathcal{R}_1(6; -2, -4)$. Corollary 5.5 gives that there are $6^2 + 4^2 - 10 = 42$ such differentials. Moreover the differential on X_2 is one of the 3 differentials of $\Omega\mathcal{M}_1(2; -2)$. Since there is a unique class of prong-matching, there exist $d_b = 42 \cdot 3 = 126$ such multi-scale differentials.

In case (c), the differential on X_1 is in one of the 35 differentials of $\Omega\mathcal{M}_1(6; -6)$. The differential on X_2 is in $\Omega\mathcal{R}_1(4; -2, -2)$. Hence Lemma 5.6 implies that there are 5 such differentials. Finally there exist $d_c = 35 \cdot 5 = 175$ such multi-scale differentials.

Summing up all the contributions, there exist

$$d = 2 \cdot (d_a + d_b + d_c) = 2 \cdot (21 + 126 + 175) = 644$$

multi-scale differentials in the closure of $\Omega\mathcal{R}_2(6; -2, -2)$ on the curve X .

To conclude that the degree of π on X is equal to 644, it remains to show that there is a unique differential which converges to each multi-scale differential. Since the family of curve is given by $xy = t$ at the node, there is a unique differential on nearby smooth curves which converges to each multi-scale differentials. \square

5.3. The class of $\overline{\mathcal{M}}_3(6; -2)$ in the Picard group of $\overline{\mathcal{M}}_3$. To compute the coefficients of boundary classes of the divisor $\overline{\mathcal{M}}_3(6, -2)$ of the projection of $\Omega\mathcal{M}_3(6, -2)$ in \mathcal{M}_3 , we use the technique of test curves. The test curves that we use are well-known and we follow the notation of [Mul17] to denote them.

First curve. The first curve A is obtained in the following way. Let X_2 be a curve of genus 2 and p a generic point of X_2 . We glue at p a base point q of a pencil of plane cubic.

Proposition 5.8. *The intersection number of $\overline{\mathcal{M}}_3(6; -2)$ with A is equal to 0.*

Proof. It is sufficient to show that there is no twisted differential of type (6; -2) on a curve semi-stably equivalent to any curve of A . Lemma 5.2 tells us that it is enough to consider cases where the points z and w are on the smooth part of X . So we have four cases to consider, which we represent in Figure 9.

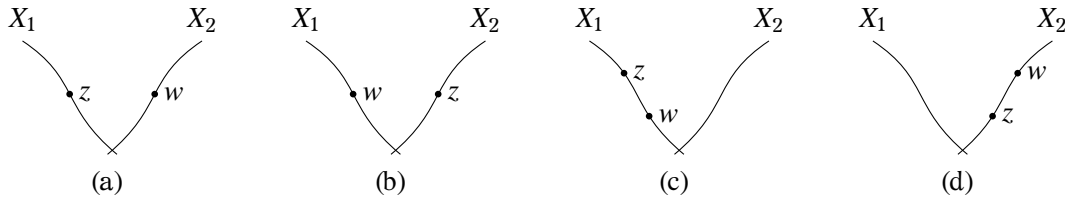


FIGURE 9. The pointed curves $(X; z, w)$ which can both underly a multi-scale differential of type (6; -2) and be an element of A .

In case (a) the restriction ω_2 of ω to X_2 is in the stratum $\Omega\mathcal{M}_2(4; -2)$. Since this stratum has a projective dimension equal to the dimension of \mathcal{M}_2 , there is a finite number of points that can be the pole of ω_2 . Hence no twisted differential of this form can exist, since the point p is generic.

In case (b), the same argument works since the differential ω_2 is in $\Omega\mathcal{M}_2(6; -4)$ with projective dimension equal to that of \mathcal{M}_2 .

In the case (c), the differential ω_2 has a zero of order 2 at p . So this point has to be a Weierstraß point. It is not possible because this point is generic on X_2 .

In the case (d), the differential ω_2 is in the stratum $\Omega\mathcal{M}_2(6; -2, -2)$. In addition, the global residue condition implies that all the residues are equal to 0. By Lemma 3.1 the locus that parametrises these differentials is of dimension three. This implies that the point p has to be special on X_2 , which gives the last contradiction. \square

Second curve. The second curve C is obtained as follows. We choose a general curve X_2 of genus 2 and an elliptic curve X_1 which we glue together at points $q_2 \in X_2$ and $q_1 \in X_1$. The family C is the family where q_2 varies over X_2 .

Proposition 5.9. *The intersection number between the divisor $\overline{\mathcal{M}}_3(6; -2)$ and the curve C is equal to 8792.*

Proof. A curve X in the family C is in the locus $\overline{\mathcal{M}}_3(6; -2)$ if and only if there is a multi-scale differential of type (6; -2) on a pointed curve semi-stably equivalent to X . Lemma 5.2 implies that no marked point can be on a rational bridge between the two components X_1 and X_2 . Moreover, the points cannot be together on a rational curve. Otherwise on this curve the differential has to be in the stratum $\Omega\mathcal{M}_0(6; -2, -6)$. By [GT21, Theorem 1.2] we know that the residues of the differential at the poles are not

zero. Then the global residue condition cannot be fulfilled. Therefore, the possible curves belonging to C and in the divisor $\overline{\mathcal{M}}_3(6; -2)$ are of the types pictured in Figure 9.

In case (a), the differential over X_1 is in the stratum $\Omega\mathcal{M}_1(6; -6)$ and the differential on X_2 is in $\Omega\mathcal{M}_2(4; -2)$. By Proposition 5.4 there are $d_2 = 2 \cdot 4^2 \cdot (-2)^2 - 18 = 110$ possible position for the points q_2 . So there are 110 curves in the intersection of C and $\overline{\mathcal{M}}_3(6; -2)$ where the zero tends to X_1 and the pole to X_2 . It remains to compute the multiplicity of these points. Note that on X_1 , there are $d_1 = 6^2 - 1 = 35$ possible non isomorphic differentials. Moreover the space of multi-scale differentials is smooth at these points. Hence this configuration contributes to $d_a = d_1 \cdot d_2 = 3850$ to the intersection number between C and $\overline{\mathcal{M}}_3(6; -2)$.

In case (b), the restriction of the multi-scale differential to X_1 is in $\Omega\mathcal{M}_1(2; -2)$ and the restriction to X_2 is in $\Omega\mathcal{M}_2(6; -4)$. So by Proposition 5.4 there are $d_2 = 2 \cdot 6^2 \cdot 4^2 - 18 = 1134$ possible positions for the point q_2 . Hence there are 1134 curves of C having multi-scale differentials of this type. Moreover, since there are $d_1 = 3$ possible differentials on X_1 and the space is smooth at these points, each of this point contributes to 3 to the intersection number. So this case contributes to $d_b = d_1 \cdot d_2 = 3402$ to the intersection number between C and $\overline{\mathcal{M}}_3(6; -2)$.

In case (c), the restriction of the multi-scale differential to X_1 is in $\Omega\mathcal{M}_1(6; -2, -4)$ and the restriction to X_2 is in $\Omega\mathcal{M}_2(2)$. So there are 6 pointed curves that have a twisted differential of this type (one for each Weierstraß point). Moreover, Corollary 5.5 tells us that there are $6^2 + 4^2 - 10 = 42$ differentials in $\Omega\mathcal{R}_1(6; -2, -4)$ over X_1 . So this case contributes to $d_b = 6 \cdot 42 = 252$ to the intersection number.

In case (d), the restriction of the multi-scale differential to X_1 is in $\Omega\mathcal{M}_1$ and the restriction to X_2 is in $\Omega\mathcal{R}_2(6; -2, -2)$. So Proposition 5.7 implies that the number of curves in C that lies in $\overline{\mathcal{M}}_3(6; -2)$ with a multi-scale differential of type (d) is 644. Since each of the intersections is simple and there are two choices for the pole of order 2, this case contributes to 1288 to the intersection number.

To conclude the proof, it is sufficient to sum the contributions of each type. Hence we have

$$\overline{\mathcal{M}}_3(6; -3) \cap [C] = 3850 + 3402 + 252 + 1288 = 8792.$$

□

Conclusion. Now it remains to collect the information of this section to give the class $\alpha\lambda + \beta\delta_0 + \gamma\delta_1$ of $\overline{\mathcal{M}}_3(6; -2)$ in $\text{Pic}(\overline{\mathcal{M}}_3) \otimes \mathbb{Q}$. The classical result of [HM98, Table 3.141] implies that:

$$\begin{aligned} \alpha + 12\beta - \gamma &= 0, \\ -2\gamma &= 8792. \end{aligned}$$

We now by Equation (11) that $\alpha = 17108$. Hence the class of $\overline{\mathcal{M}}_3(6; -2)$ in $\text{Pic}(\overline{\mathcal{M}}_3) \otimes \mathbb{Q}$ is given by

$$(12) \quad [\overline{\mathcal{M}}_3(6; -2)] = 17108\lambda - 1792\delta_0 - 4396\delta_1.$$

To conclude note that [Boi15] gives that $\Omega\mathcal{M}_3(6; -2)$ has one hyperelliptic components and two other components distinguished by parity. We denote by $\mathcal{M}_3^{\text{hyp}}(6; -2)$ the projection of the hyperelliptic component and by $\mathcal{M}_3^+(6; -2)$ and $\mathcal{M}_3^-(6; -2)$ the projection of the other components. In the hyperelliptic case the zero and the pole are distinct Weierstraß points. Since there are 8 such points, the class of $\mathcal{M}_3^{\text{hyp}}(6; -2)$ is 56 times the class of the hyperelliptic divisor in $\overline{\mathcal{M}}_3$. According to [HM98, Equation 3.165], the class of the hyperelliptic locus is $9\lambda - \delta_0 - 3\delta_1$. Hence a direct consequence of Theorem 1.1 is the following result.

Corollary 5.10. *In the rational Picard group of the stack $\overline{\mathcal{M}}_3$ we have*

$$\left[\overline{\mathcal{M}}_3^+(6; -2)\right] + \left[\overline{\mathcal{M}}_3^-(6; -2)\right] = 16604\lambda - 1736\delta_0 - 4228\delta_1.$$

From this corollary we obtain that the slope of the divisor $\overline{\mathcal{M}}_3^+(6; -2) + \overline{\mathcal{M}}_3^-(6; -2)$ is equal to $9 + \frac{35}{62}$. Note that this is coherent with the fact [HM90, Corollary 0.5] that the only divisor of slope less than $\frac{28}{3}$ in $\overline{\mathcal{M}}_3$ is the divisor of hyperelliptic curves.

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